# $\boxed{\text{Lecture 1: Superconducting grains}}$

## a) Bardeen-Cooper-Schrieffer theory (1957)

- $$H = \int dr \left\{ \underbrace{\sum_\sigma \psi^+_\sigma(r)\left(\frac{-\nabla^2}{2m}\right)\psi_\sigma(r)}_{\text{free fermions}} \underbrace{- |\lambda| V n_\uparrow(r) n_\downarrow(r)}_{\text{local attraction}} \right\}$$

$$\psi_\sigma(r) = \frac{1}{\sqrt{V}} \sum_k a_{k\sigma} e^{ik.r} \qquad \text{spin } \sigma = \begin{cases} \uparrow \text{ or } + \\ \downarrow \text{ or } - \end{cases} \qquad \varepsilon_k = \frac{k^2}{2m}$$

$$H = \sum_{k\sigma} \varepsilon_k a^+_{k\sigma} a_{k\sigma} - |\lambda| \sum_{kk'q} a^+_{k+q\uparrow} a^+_{k'-q\downarrow} a_{k'\downarrow} a_{k\uparrow}$$

fermions: $\{a_{k\sigma}, a_{k'\sigma'}\} = 0$, $\{a_{k\sigma}, a^+_{k'\sigma'}\} = \delta_{kk'}\delta_{\sigma\sigma'}$

- mean-field decoupling in grand canonical ensemble:

$$a_{k'\downarrow} a_{k\uparrow} = \underbrace{\langle a_{-k\downarrow} a_{k\uparrow}\rangle}\delta_{k,-k'} + \underbrace{a_{k'\downarrow} a_{k\uparrow} - \langle a_{-k\downarrow} a_{k\uparrow}\rangle \delta_{k,-k'}}_{\text{assumed to be small}}$$

- quantum-statistical average
- vanishes in canonical ensemble (number of particles is conserved)
- interpretation will come later

$$H - \mu N = \sum_{k\sigma} \xi_k a^+_{k\sigma} a_{k\sigma} + \Delta \sum_k a^+_{k\uparrow} a^+_{-k\downarrow} + \Delta^* \sum_k a_{-k\downarrow} a_{k\uparrow} + \frac{|\Delta|^2}{|\lambda|}$$

with $\Delta = -|\lambda| \sum_k \langle a_{-k\downarrow} a_{k\uparrow}\rangle$ and $\xi_k = \varepsilon_k - \mu$

$\longrightarrow$ quadratic form: exact diagonalization

- Bogoliubov transformation:

$$H - \mu N = \sum_k \left(a^+_{k\uparrow} \;\; a_{-k\downarrow}\right) \begin{pmatrix} \xi_k & \Delta \\ \Delta^* & -\xi_k \end{pmatrix} \begin{pmatrix} a_{k\uparrow} \\ a^+_{-k\downarrow} \end{pmatrix} + \sum_k \xi_k + \frac{|\Delta|^2}{|\lambda|}$$

$$\downarrow$$

$$\mathcal{H} = \xi_k \tau_3 + |\Delta|(\cos\varphi\, \tau_1 - \sin\varphi\, \tau_2)$$
$$= e^{i\frac{\varphi}{2}\tau_3}\left[\xi_k \tau_3 + |\Delta| \tau_1\right] e^{-i\frac{\varphi}{2}\tau_3}$$
$$= e^{i\frac{\varphi}{2}\tau_3} e^{-i\frac{\theta}{2}\tau_2} E_k \tau_3 e^{+i\frac{\theta}{2}\tau_2} e^{-i\frac{\varphi}{2}\tau_3}$$

with $\boxed{E_k = \sqrt{\xi_k^2 + |\Delta|^2}}$

$$\tan\theta_k = \frac{|\Delta|}{\xi_k} \quad \text{and} \quad 0 < \theta_k < \pi$$

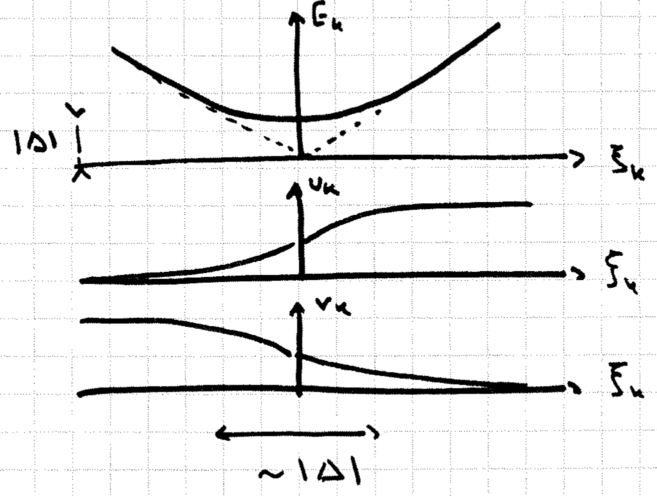
$$H - \mu N = \sum_k \left(\alpha^+_{k\uparrow} \;\; \alpha_{-k\downarrow}\right) \begin{pmatrix} E_k & 0 \\ 0 & -E_k \end{pmatrix} \begin{pmatrix} \alpha_{k\uparrow} \\ \alpha^+_{-k\downarrow} \end{pmatrix} + \sum_k \xi_k + \frac{|\Delta|^2}{|\lambda|}$$

with $\begin{pmatrix} \alpha_{k\uparrow} \\ \alpha^+_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & +v_k \\ -v_k & u_k \end{pmatrix} \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix} \begin{pmatrix} a_{k\uparrow} \\ a^+_{-k\downarrow} \end{pmatrix}$

$\hookrightarrow \boxed{a_{k\sigma} = e^{i\frac{\varphi}{2}}\left(u_k \alpha_{k\sigma} - \sigma v_k \alpha^+_{-k-\sigma}\right)}$

$$\boxed{\begin{aligned} u_k &= \cos\frac{\theta_k}{2} = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{\xi_k}{E_k}} \\ v_k &= \sin\frac{\theta_k}{2} = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{\xi_k}{E_k}} \end{aligned}}$$

or $\alpha_{k\sigma} = e^{-i\frac{\varphi}{2}} u_k a_{k\sigma} + \sigma e^{i\frac{\varphi}{2}} v_k a_{-k-\sigma}$




$\xi_k \gg |\Delta|$ $\qquad \alpha_{k\sigma} \approx a_{k\sigma}$

"particle"

$\xi_k \ll -|\Delta|$ $\qquad \alpha_{k\sigma} \approx -\sigma\, a_{-k-\sigma}$

"hole"

- Ground state $\quad |BCS_\varphi\rangle = \prod_k \left(u_k - e^{i\varphi} v_k\, a^+_{k\uparrow} a^+_{-k\downarrow}\right)|vac.\rangle$

such that $\quad \alpha_{k\sigma}|BCS_\varphi\rangle = 0$

<u>example:</u> $\alpha_{k\uparrow}|BCS_\varphi\rangle = \prod_{k'\neq k}\left(u_{k'} - e^{i\varphi} v_{k'}\, a^+_{k'\uparrow} a^+_{-k'\downarrow}\right)\left(e^{-i\frac{\varphi}{2}} u_k\, a_{k\uparrow} + e^{i\frac{\varphi}{2}} v_k\, a^+_{-k\downarrow}\right)\left(u_k - e^{i\varphi} v_k a^+_{k\uparrow} a^+_{-k\downarrow}\right)|vac.\rangle$

$$\propto \prod_{k'\neq k}\dots \times \left(u_k v_k e^{i\frac{\varphi}{2}} a^+_{-k\downarrow} - u_k v_k e^{i\frac{\varphi}{2}} \underbrace{a_{k\uparrow} a^+_{k\uparrow}}_{1} a^+_{-k\downarrow}\right)|vac\rangle$$

$$= 0$$

* $\Delta = 0$ $\qquad |BCS_\varphi\rangle \propto \prod_{|k|<k_F,\sigma} a^+_{k\sigma}|vac.\rangle$ is the Fermi sea

(Normal state)

[ $\varphi$ absorbed in a global phase ]

* $\Delta \neq 0$ $\qquad$ correlated pairs with <u>opposite</u> spins & momenta

(superconducting state)

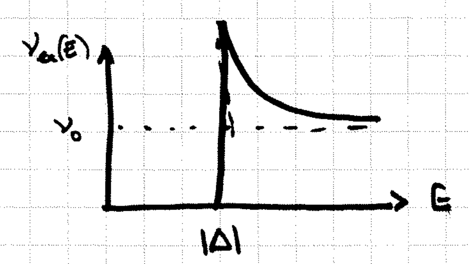
Cooper pairs condense in ground state

↳ superconductivity is destroyed by a magnetic field through orbital or Zeeman effects

- Excitation spectrum is gapped:

density of states per spin: $\quad \nu_{ex}(E) = \sum_k \delta(E - E_k) = 2\nu_0 \dfrac{E}{\sqrt{E^2 - |\Delta|^2}}\,\theta(E - |\Delta|)$

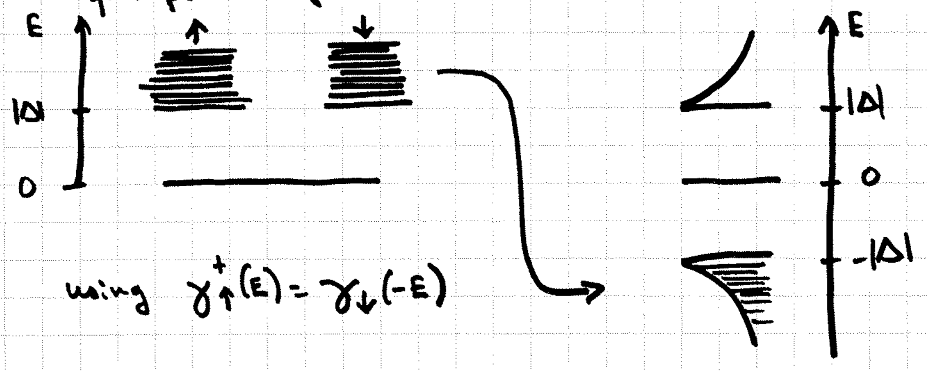
$\nu_0 = \dfrac{m k_F}{2\pi^2}$ in 3D




<u>NB:</u> $\int dE\, \nu_S(E) = \int dE\, \nu_N(E)$ is conserved

⚠ two equivalent representations in literature:

quasiparticle picture: $\qquad$ semiconductor picture:




using $\gamma^+_\uparrow(E) = \gamma_\downarrow(-E)$

gap $\Delta$ by injecting/extracting electrons (N/I/S junctions)

gap $2\Delta$ by breaking a Cooper pair into two excitations (optics, S/I/S junctions)

- selfconsistency equation:

$$\Delta = -|\lambda| \sum_k \langle (u_k \alpha_{-k\downarrow} + v_k \alpha^+_{k\uparrow})(u_k \alpha_{k\uparrow} - v_k \alpha^+_{-k\downarrow}) \rangle e^{i\varphi}$$

$$= -|\lambda| \sum_k \underbrace{u_k v_k e^{i\varphi}}_{\frac{\Delta}{2E_k}} \left[ f_k - (1 - f_k) \right]$$

$\langle \alpha^+_{k\sigma} \alpha_{k\sigma} \rangle = f(E_k)$

Fermi function $f(E) = \frac{1}{2}\left(1 - \text{th}\,\frac{E}{2T}\right)$

$$1 = \frac{|\lambda|}{2} \sum_k \frac{1}{E_k} \text{th}\,\frac{E_k}{2T} = \frac{\nu_0 |\lambda|}{2} \int \frac{d\xi}{E} \text{th}\,\frac{E}{2T} \qquad g = |\lambda| \nu_0$$

* $T \to 0$: $\quad 1 = g \int_0^{\Omega} \frac{d\xi}{\sqrt{\xi^2 + \Delta_0^2}}$ cut-off to regularize a log-divergency

$$= g \left[ \ln\left[\xi + \sqrt{\xi^2 + \Delta(0)^2}\right] \right]_0^{\Omega}$$

$$= g \ln \frac{2\Omega}{\Delta(0)} \longrightarrow \boxed{\Delta(0) = 2\Omega e^{-\frac{1}{g}}}$$

* $T \to T_c$: $\Delta \to 0$

$$1 = g \int_0^{\Omega_0} \frac{d\xi}{\xi} \text{th}\,\frac{\xi}{2T_c} = g \int_0^{\Omega_0/2T_c} \frac{dx}{x} \text{th}\,x$$

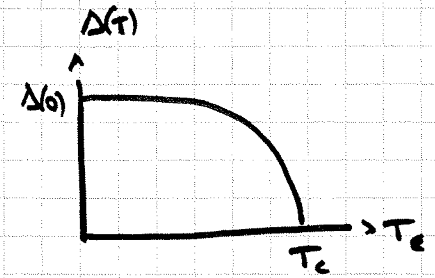
$$= g \left\{ \left[ \text{th}\,\xi \ln \xi \right]_0^{\Omega/2T_c} - \int_0^{\Omega_0/2T_c} d\xi \frac{\ln \xi}{\text{ch}^2 x} \right\}$$

converges, $\frac{\Omega_0}{2T_c} \to \infty$

$$\approx g \left\{ \ln \frac{\Omega_0}{2T_c} - \ln \frac{\pi}{4\gamma} \right\}$$

$\boxed{T_c = 1.14\, \Omega e^{-\frac{1}{g}}}$

$\boxed{\frac{\Delta(0)}{T_c} = 1.76} \; \left(= \frac{\pi}{\gamma}\right)$




second order phase transition

$\Delta(T) \sim \sqrt{T_c(T_c - T)} \qquad T \lesssim T_c$

- $H - \mu N = \sum_{k\sigma} E_k \alpha^+_{k\sigma} \alpha_{k\sigma} + \sum_k (E_k - \xi_k) + \frac{|\Delta|^2}{|\lambda|}$

at $T = 0$: $\quad \Delta E = -\sum_k \left\{ E_k - \xi_k - \frac{\Delta^2}{2E_k} \right\}$

$$= -2\nu_0 \int_0^{\Omega} d\xi \left[ \sqrt{\xi^2 + \Delta^2} - \xi - \frac{\Delta^2}{2\sqrt{\xi^2 + \Delta^2}} \right]$$

converges!

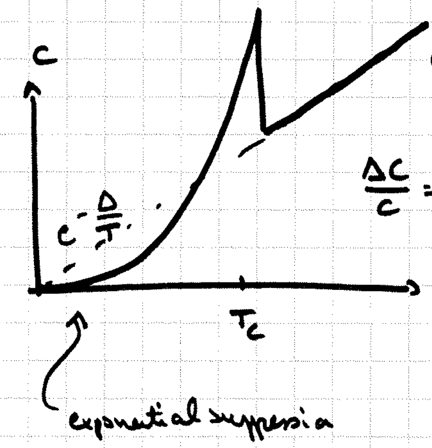
$$= -\frac{\nu_0 \Delta^2}{2}$$

condensation energy

binding energy $\Delta$ gained for a fraction $\nu\Delta$ of the conduction electrons.

At finite temperature : free energy $\mathcal{F}(T)$

$\hookrightarrow$ specific heat $\quad C = -T \dfrac{d^2\mathcal{F}}{dT^2}$




$C_N = \dfrac{2\pi^2}{3}\nu_0 T$

$\dfrac{\Delta C}{C} = \dfrac{12}{7\zeta(3)} \approx 1.42$

b) Fluctuations

[redacted]

- Cooper pair size $\quad \Delta \sim v\,\delta p \sim \dfrac{v}{\xi_{bal}} \qquad \xi_{bal} \sim \dfrac{\hbar v_F}{\Delta} \gg \lambda_F$

$\hookrightarrow$ in diffusive systems with a mean free path $\ell$, the superconducting coherence length is reduced:

$$\left.\begin{aligned} \mathcal{L} &= v_F t \\ L &= \sqrt{Dt} \end{aligned}\right\} \quad L \sim \sqrt{\ell \mathcal{L}}$$

$$\xi_{diff} \sim \sqrt{\ell\, \xi_{bal}} \sim \sqrt{\frac{D}{\Delta}}$$ [redacted]

$\hookrightarrow$ scale over which variations of the order parameter weaken superconductivity

- Free energy functional (Ginzburg-Landau theory, 1950)

$\Delta = \Delta_0 \times \psi$

$$\mathcal{F} \sim \underbrace{\nu_d \Delta^2}_{\text{condensation energy per unit volume}} \int dx^d \left\{ \frac{T-T_c}{T_c} |\psi|^2 + \xi^2 |\nabla \psi|^2 + |\psi|^4 \right\}$$

near $T_c$

partition function: $\quad Z \sim \int \mathcal{D}\psi \, e^{-\frac{\mathcal{F}[\psi]}{T}}$

extremalization of the functional: $\quad \dfrac{\delta \mathcal{F}}{\delta \psi} = 0 \longrightarrow$ Ginzburg-Landau equations

in particular: [redacted] $\psi \sim \begin{cases} 0 & T > T_c \\ \sqrt{\frac{T_c - T}{T_c}} & T < T_c \end{cases}$

- Justification for the saddle point approximation:

$$\left.\begin{aligned} \psi &\sim \sqrt{\frac{|T_c - T|}{T_c}}\,\psi' \\ x &\sim \xi \sqrt{\frac{T_c}{|T_c - T|}}\, x' \end{aligned}\right\} \quad \frac{\mathcal{F}}{T_c} \sim \underbrace{\nu_d \Delta\, \xi^d \left(\frac{|T - T_c|}{T_c}\right)^{\frac{2-d}{2}}}_{\text{must be large!}} \underbrace{\int dx' \left[ \pm |\psi'|^2 + |\nabla \psi'|^2 + |\psi'|^4 \right]}_{\text{dimensionless and scaleless}}$$

↳ the mean-field theory works in the vicinity of the critical temperature

for $\frac{|\delta T|}{T_c} \gg Gi$ where $Gi \sim \left(\frac{1}{\nu_d \Delta \xi^d}\right)^{\frac{1}{2-\frac{d}{2}}}$

examples: * clean 3d metal: $Gi \sim \left[\frac{1}{m k_F \Delta \left(\frac{v_F}{\Delta}\right)^3}\right]^2 \sim \left(\frac{T_c}{\epsilon_F}\right)^4$ very small!

* diffusive 2d metal: $Gi \sim \frac{1}{m k_F t \Delta \cdot \frac{D}{\Delta}} \sim \frac{1}{\nu_0 D_n t} = \frac{e^2/\hbar}{\sigma d} = \frac{G_Q}{G_\square}$

film thickness

inverse of the conductance per square

* superconducting grain (0d)

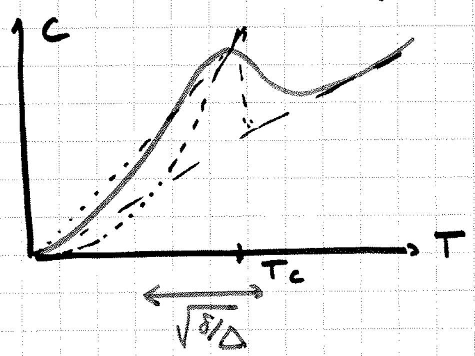
$$Gi \sim \left(\frac{1}{\nu L^3 \Delta}\right)^{\frac{1}{2}} \sim \sqrt{\frac{\delta}{\Delta}}$$ ↳ mean level spacing

* Superconductivity is insensitive to moderate disorder [Anderson Theorem]

* It persists in grains of size $L \ll \frac{1}{(\nu \Delta)^{1/3}} \sim (\lambda_F^2 \xi)^{1/3} \ll \xi$ much smaller than $\xi$

* Destruction by a strong disorder $\Delta \sim \delta_{t_{loc}}$ ?

in 0d




c) Parity effect

- Back to canonical ensemble:

$$|BCS_N\rangle = \int \frac{d\varphi}{2\pi} e^{-iN\varphi} |BCS_\varphi\rangle$$

for $N$ even (N number of electrons)

and similarly for excitations: $\int \frac{d\varphi}{2\pi} e^{-iN\varphi} \left(\gamma^+_{k\sigma}(\varphi) |BCS_\varphi\rangle\right)$ for N odd

- $T \ll \Delta$ even state: all electrons are paired into Cooper pairs $Z_e \approx 1$

odd state: 1 unpaired electron

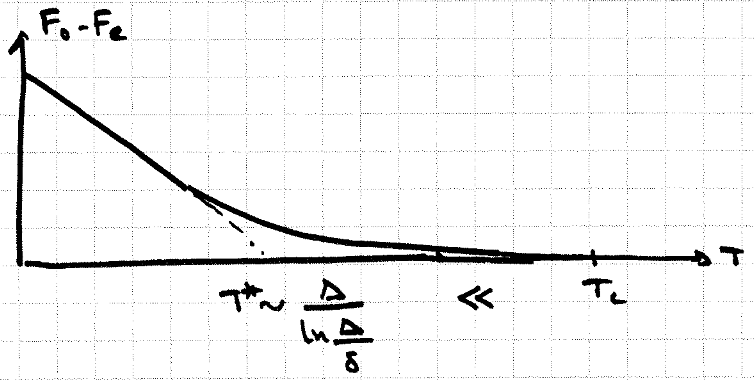
$$Z_o = \sum_{k\sigma} e^{-\frac{E_k}{T}} = \frac{2}{\delta}\int d\xi\, e^{-\frac{\sqrt{\xi^2+\Delta^2}}{T}} \approx \frac{2}{\delta} e^{-\frac{\Delta}{T}} \int d\xi\, e^{-\frac{\xi^2}{2\Delta T}} \approx \frac{\sqrt{8\pi T\Delta}}{\delta}$$

$$F_{e/o} = -T \ln Z_{e/o}$$

$$F_o - F_e = \Delta - T \ln \frac{\sqrt{8\pi T \Delta}}{\delta}$$

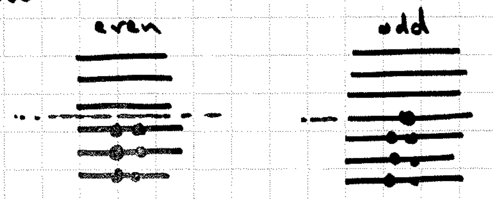
energy cost

entropy gain




* parity effect if $T \ll T^*$
* $T^* \ll T_c$ and also controlled by the ratio $\delta/\Delta$
* We assumed that $\Delta$ was not affected by parity effects:

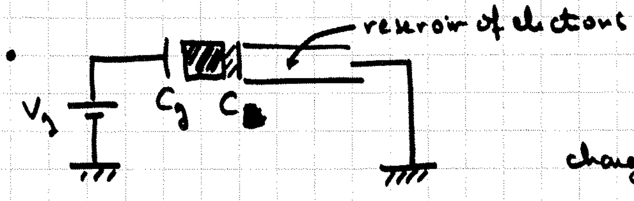
check: model of equispaced levels in usual state




$$\sum_n \frac{1}{\sqrt{(n+\frac{1}{2})^2\delta^2+\Delta_e^2}} = \sum_n \frac{1}{\sqrt{n^2\delta^2+\Delta_o^2}} \Rightarrow \Delta_o \approx \Delta_e - \underbrace{\delta/2}_{\text{small}}$$

d) Coulomb blockade

•




$q = -en$ charge in the island

charging energy:

$$E_{el}(q+\delta q) - E_{el}(q) = \underbrace{\delta\left(\frac{q^2}{2C_\Sigma}\right)}_{\text{electrostatic energy}} + \underbrace{V_g \frac{C_g}{C_\Sigma}\delta q}_{\text{work of the battery}}$$

$C_\Sigma = C_g + C$

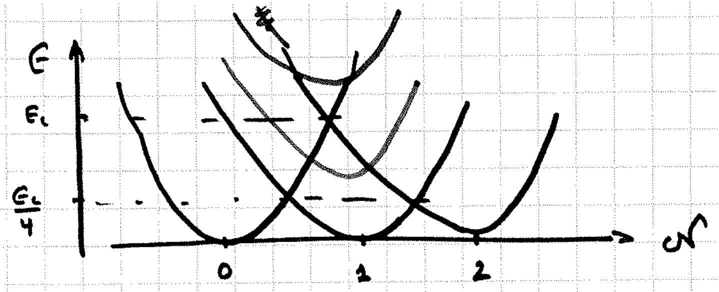
$$\Rightarrow E_{el}(q) = \frac{1}{2C_\Sigma}(q + C_g V_g)^2 + \text{cste}$$

• superconducting island at $T \ll T^*$

$$E(n) = E_c(n-\mathcal{N})^2 + \frac{1-(-)^n}{2}\Delta$$

$E_c = \dfrac{e^2}{2C_\Sigma}$

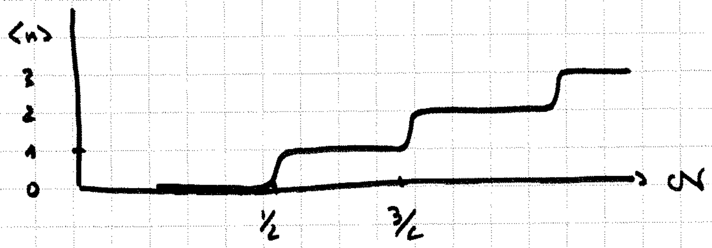
$\mathcal{N} = C_g V_g$

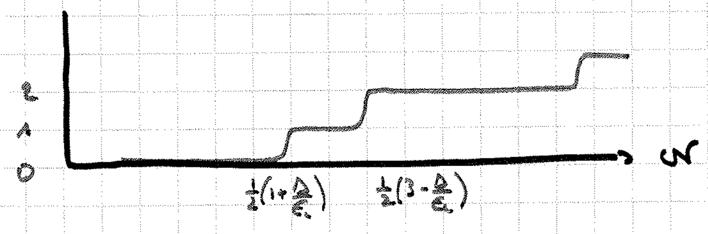



$\Delta = 0$

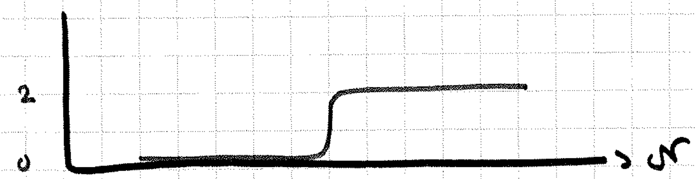
$\Delta < E_c$

$\Delta > E_c$




long even and short odd steps

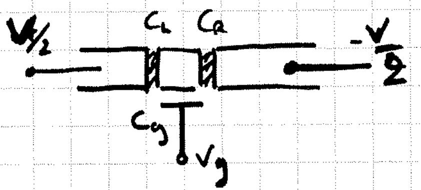



only even steps

NB: The Coulomb staircase has sharp steps if

$$\begin{cases} T \ll E_c \\ \frac{1}{\tau_{RC}} \ll E_c \quad \longleftrightarrow \quad R \gg \hbar/e^2 \end{cases}$$

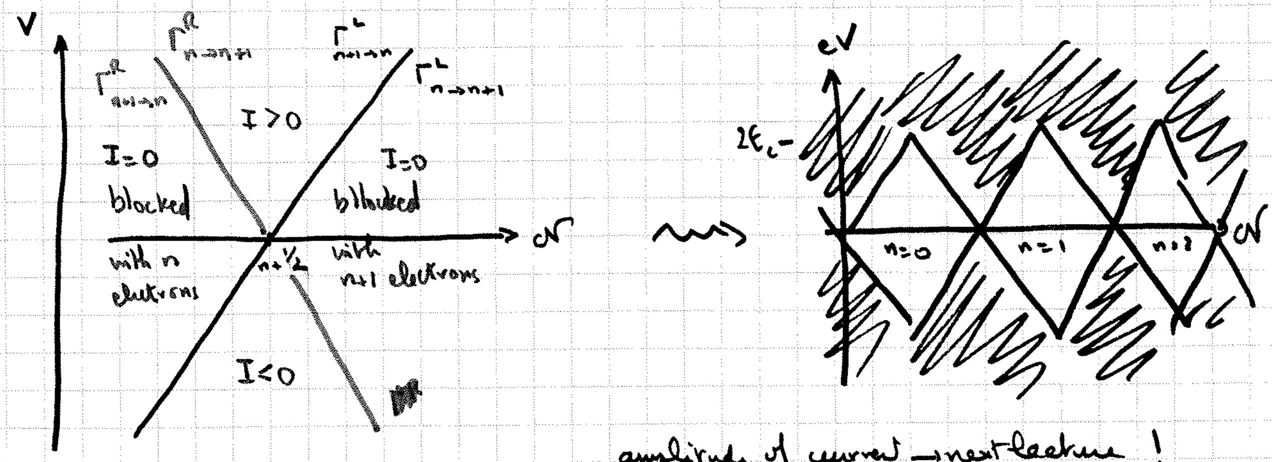
- transport:




for simplicity, $C_L = C_R$

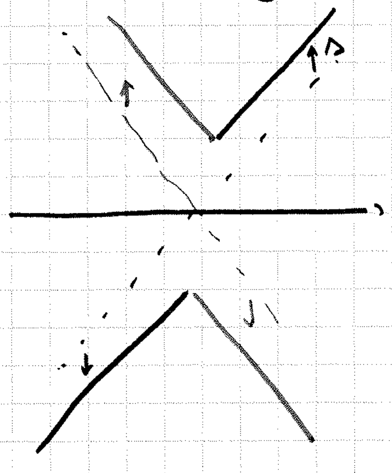
→ in N state, an electron can jump from left/right lead to the island if

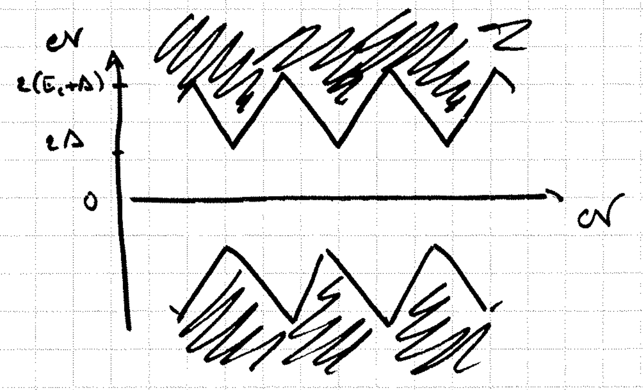
$$\mp \frac{eV}{2} + E(n) > E(n+1) = E(n) - 2E_c\left(n_g - (n+\tfrac{1}{2})\right)$$




amplitude of current → next lecture !

- In N/I/S/I/N junctions, creating a quasiparticle in S costs energy $\Delta$:

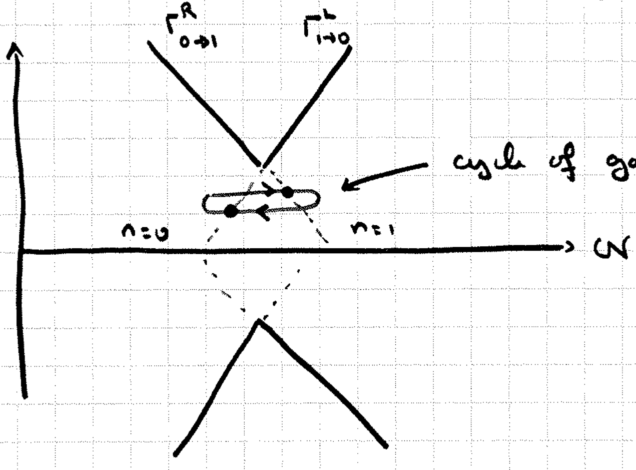



e-periodic structure

(at $T \ll T^*$ parity effects yield a 2e-periodic structure)

- Turnstile (Pekola et al. 2008)




cycle of gate voltage

$\boxed{I = ef}$

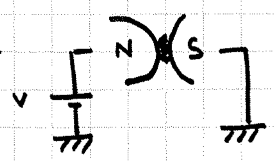
NB: works also in SINIS (and better)

**Lecture 2** Quasiparticle and Andreev tunneling in normal/superconductor junctions

a) quasiparticle current

$(e>0)$

- setup:




$$H = H_L + H_R + H_T$$
$$= \sum_{k\sigma} (\xi_k - eV)\, a^+_{k\sigma} a_k + \sum_{p\sigma} \varepsilon_p \beta^+_{p\sigma}\beta_{p\sigma} + \sum_{kp\sigma} t_{kp}\, a^+_{k\sigma} b_{p\sigma} + h.c.$$
$$b_{p\sigma} = e^{i\frac{\varphi}{2}}\left[u_p \beta_{p\sigma} - \sigma v_p \beta^+_{-p-\sigma}\right]$$

For simplicity, we assume $t_{kp} \simeq t_0$ (i.e. independent of $k$ and $p$ for momenta close to Fermi wavevector), which corresponds to a point-contact

- The rates to transfer electrons through the junction can be obtained with the Fermi Golden Rule:

$$\Gamma_{R\to L} = 2\pi |t_0|^2 \sum_{kp\sigma}\left[u_p^2 f_p (1-f_k)\, \delta(\xi_k - eV - \varepsilon_p) + v_p^2\, f_k\, f_p\, \delta(\xi_k - eV + \varepsilon_p)\right]$$
$$\Gamma_{L\to R} = 2\pi |t_0|^2 \sum_{kp\sigma}\left[u_p^2 (1-f_p) f_k\, \delta(\xi_k - eV - \varepsilon_p) + v_p^2 (1-f_k)(1-f_p)\, \delta(\xi_k - eV + \varepsilon_p)\right]$$

The current is

$$I = e\left(\Gamma_{R\to L} - \Gamma_{L\to R}\right) \qquad + \text{spin degeneracy}$$
$$= 4\pi e |t_0|^2 \underbrace{\sum_{kp}}_{\int_0^{\infty} dE\, \delta(E-\varepsilon_p) \int_{-\infty}^{+\infty} d\xi\, \delta(\xi - \xi_k)} \left[u_p^2 (f_p - f_k)\, \delta(\xi_k - eV - \varepsilon_p) - v_p^2 (1 - f_k - f_p)\, \delta(\xi_k - eV + \varepsilon_p)\right]$$

$$u_p^2, v_p^2 = \frac{1}{2}\left(1 \pm \frac{\xi_p}{\varepsilon_p}\right) \quad \hookrightarrow \text{cancels with } \sum_p$$

$$= 4\pi e |t_0|^2 \int_0^{\infty} dE \int_{-\infty}^{+\infty} d\xi\; \nu_N\, \nu_{ex}(E) \left[\left[f(E) - f(\xi)\right]\delta(\xi - eV - E) - \left(1 - f(\xi) - f(E)\right)\delta(\xi - eV + E)\right]$$

$$\xi, \; E \to -E$$

$$= 4\pi e |t_0|^2 \int_{-\infty}^{+\infty} dE \int_{-\infty}^{+\infty} d\xi\; \nu_N\, \nu_S(E) \left[f(E) - f(\xi)\right] \delta(\xi - E - eV)$$

with $\nu(E) = \nu_S \dfrac{|E|}{\sqrt{E^2 - \Delta^2}}\, \Theta(|E| - \Delta)$

convolution of densities of states

ponderated with Fermi functions

$$I = \frac{G_N}{e} \int_{|E|>\Delta} dE\, \frac{|E|}{\sqrt{E^2 - \Delta^2}} \left[f(E) - f(E + eV)\right] \qquad \text{with } G_N = 4\pi e^2 |t_0|^2 \nu_L \nu_R$$

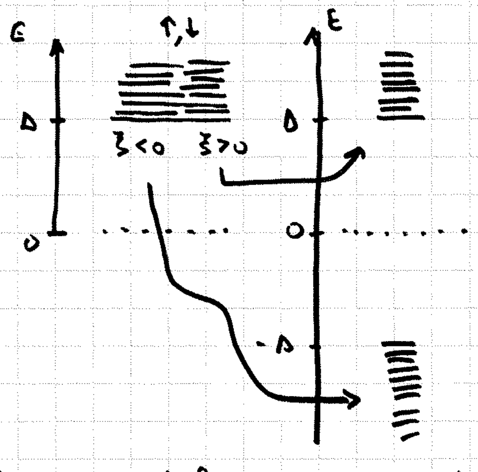
N state: $I = G_N V$

S state: $I = G_N \sqrt{V^2 - \Delta^2/e^2}$ at $T = 0$, $eV > \Delta$ (gap)

$$\left.\frac{\partial I}{\partial V}\right|_{V\to 0} = \frac{G_N}{e} \int_{|E|>\Delta} dE\, \frac{|E|}{\sqrt{E^2 - \Delta^2}}\, \frac{1}{4T\,\mathrm{ch}^2\frac{E}{2T}}$$

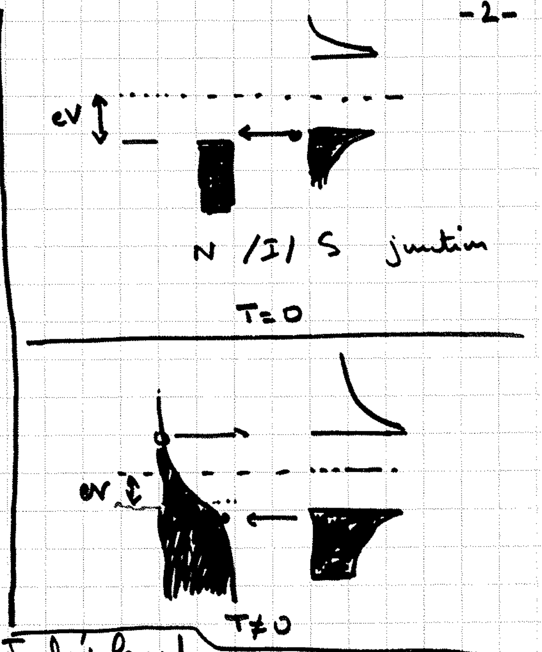
$$\propto e^{-\Delta/T} \;\longrightarrow\; \text{thermometer}$$

$$\left.\frac{\partial I}{\partial V}\right|_{T\to 0} \propto \nu_S(eV) \;\longrightarrow\; \text{tunnel spectroscopy (STM)}$$




quasiparticle picture

semiconductor picture




• thermal transport:

the rates for extracting energy out of the reservoirs can also be calculated with Fermi Golden rule:

$$\Gamma^{heat}_{out\,of\,N} = 4\pi t_0^2 \sum_{kp} \xi_k \Big\{ u_p^2\big[f_k(1-f_p) - f_p(1-f_k)\big]\,\delta(\xi_k - eV - \varepsilon_p) + v_p^2\big[f_k f_p - (1-f_k)(1-f_p)\big]\,\delta(\xi_k - eV + \varepsilon_p)\Big\}$$

$$\Gamma^{heat}_{out\,of\,S} = 4\pi t_0^2 \sum_{kp} \varepsilon_p \Big\{ u_p^2\big[f_p(1-f_k) - f_k(1-f_p)\big]\,\delta(\xi_k - eV - \varepsilon_p) + v_p^2\big[f_k f_p - (1-f_k)(1-f_p)\big]\,\delta(\xi_k - eV + \varepsilon_p)\Big\}$$

one can check that $\Gamma^{heat}_{out\,of\,N} + \Gamma^{heat}_{out\,of\,S} = - V \times I(V)$ Joule's law! $\}$ heating

thus $\Gamma^{heat}_{out\,of\,L} = \Gamma^{heat}_{out\,of\,R} = -\frac{1}{2} I V < 0$ in each N reservoir.

But, the situation is different in a N/I/S junction:

let us assume $T \ll \Delta - eV \ll eV$, then

$$\Gamma^{heat}_{out\,of\,N} \simeq \frac{G_N}{e^2} \int_{|E|>\Delta} dE \frac{|E|}{\sqrt{E^2-\Delta^2}} (E+eV)\big[f(E+eV) - f(E)\big]$$

$$\approx \frac{G_N}{e^2} \int_{-\infty}^{-\Delta} dE \frac{\Delta}{\sqrt{2\Delta(-E-\Delta)}} (-\Delta+eV)\left(1 - e^{-\frac{E+eV}{T}} - 1\right)$$

$$\approx \frac{G_N}{e^2} \sqrt{\frac{\Delta}{2}}\,(\Delta - eV)\, e^{-\frac{\Delta-eV}{T}} \underbrace{\int_0^\infty \frac{dx}{\sqrt{x}} e^{-\frac{x}{T}}}_{} \qquad x = T y^2$$

$$= \frac{G_N}{e^2} \frac{\sqrt{\pi T \Delta}}{2} (\Delta - eV)\, e^{-\frac{\Delta - eV}{T}} > 0 \;!$$

heat is extracted from N $\Rightarrow$ N is cooled!

b) Andreev reflection

- At $T, eV \ll \Delta$, the quasiparticle current is small

However, Bogoliubov quasiparticles may exist as virtual excitations

$2^{nd}$ order perturbation theory in tunnel Hamiltonian:

$$H = H_N + \underbrace{H_S}_{\text{qps with energy} \gtrsim \Delta} + H_T$$

$$H_{eff} \simeq H_N - H_T \frac{1}{H_S + \underbrace{H_N}_{\text{good approximation for qps in N with energy} \ll \Delta}} H_T$$

$$H_T = t_0 \sum_{kp\sigma} a^+_{k\sigma} \left[ u_p \beta_{p\sigma} - \sigma v_p \beta^+_{-p,-\sigma} \right] e^{i\frac{\varphi}{2}} + h.c.$$

$$= t_0 \sum_{kp\sigma} \beta^+_{p\sigma} \left[ u_p e^{-i\varphi/2} a_{k\sigma} + \sigma v_p e^{+i\frac{\varphi}{2}} a^+_{-k,-\sigma} \right] + h.c.$$

$$H_{eff} \simeq H_N - t_0^2 \sum_{kk'p\sigma} \left[ u_p e^{+i\frac{\varphi}{2}} a^+_{k'\sigma} + \sigma v_p e^{-i\frac{\varphi}{2}} a_{-k',-\sigma} \right] \frac{1}{E_p} \left[ u_p e^{-i\varphi/2} a_{k\sigma} + \sigma v_p e^{i\frac{\varphi}{2}} a^+_{-k\sigma} \right]$$

$$\simeq H_N - t_0^2 \sum_{kk'\sigma} \left\{ \underbrace{\sum_p \left( \frac{u_p^2 - v_p^2}{E_p} \right)}_{=0} a^+_{k'\sigma} a_{k\sigma} + \sigma \underbrace{\left( \sum_p \frac{u_p v_p}{E_p} \right)}_{= \nu_S \int d\xi \frac{\Delta}{2E^2} = \frac{\pi \nu_S}{2}} e^{i\varphi} a^+_{k'\sigma} a^+_{k,-\sigma} + h.c. \right\}$$

$= 0$ i.e. no additionnal potential scattering is generated

looks like a local pairing term!

$$H_{eff} \simeq H_N - \pi \nu_S t_0^2 \sum_{kk'} a^+_{k\uparrow} a^+_{k'\downarrow} e^{i\varphi} + h.c.$$

- We can now use Fermi Golden Rule to calculate the rate for creation of two quasiparticles in N:

$$\Gamma^A_{R\to L} = 2\pi (\pi \nu_S t_0^2)^2 \sum_{kk'} (1-f_k)(1-f_{k'})\, \delta(\xi_k + \xi_{k'} - 2eV)$$

$$\Gamma^A_{L\to R} = \text{―――――――} \quad f_k f_{k'}$$

$$I_A = (2e)\left(\Gamma_{R\to L} - \Gamma_{L\to R}\right)$$

$$= 4\pi e\, (\pi \nu_S t_0^2)^2 \nu_N^2 \int d\xi d\xi' \left[1 - f(\xi) - f(\xi')\right] \delta(\xi + \xi' - 2eV)$$

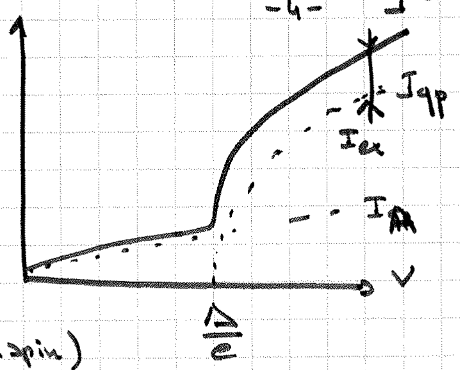
$$= G_A V$$

Andreev conductance:

$$G_A = 8\pi e^2 (\pi \nu_S \nu_N t_0^2)^2 = \frac{\pi G_N^2}{2 e^2}$$

→ Subgap current

→ excess current $I - G_N V\big|_{V \gg \Delta/e} \neq 0$

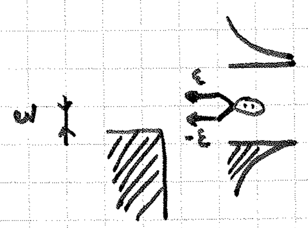



Andreev reflection: (AR)

1 Cooper pair → two electrons with opposite energies (and spin) with respect to the chemical potential in S

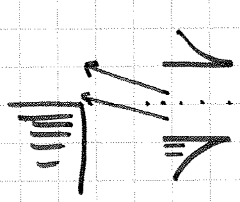
or equivalently:

1 incident electron is reflected as a hole in N, while a Cooper pair is created



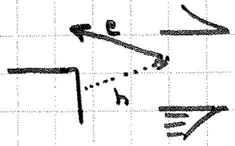

(works if $|\varepsilon| < eV$)

⟶ with slopes for electron transfer



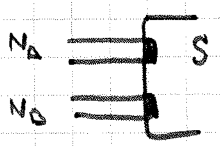
slope $-eV$ for electrons

⇕




slope $-eV$ for electrons
$+eV$ for holes

c) multiterminal geometry




Same approach

$$H = H_A + H_B + \sum_{\substack{kk' \\ \alpha = A,B}} \left( t_{\alpha\beta}\, a^+_{k\alpha\uparrow} a^+_{k'\beta\downarrow} + h.c. \right) + \sum_{kk'\sigma} \left( t'_{AB}\, a^+_{kA\sigma} a_{k'B\sigma} + h.c. \right)$$

direct and crossed AR $\qquad$ "elastic cotunneling"

$t_{\alpha\beta} \sim \cos^2 k_F x_{\alpha\beta}\, e^{-\frac{x_{\alpha\beta}}{\xi_s}}$ $\qquad$ $t'_{AB} \sim \sin^2 k_F x_{AB}\, e^{-\frac{x_{AB}}{\xi_s}}$

↳ production of entangled pairs of electrons

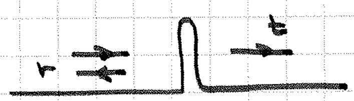
$$I_A = G_A V_A + G_{CAR}(V_A + V_B) + G_{EC}(V_A - V_B)$$

$$G_{nl} = \frac{\partial I_A}{\partial V_B} = G_{CAR} - G_{EC}$$

NB: $\langle \cos^2 k_F x_{AB} \rangle \approx \langle \sin^2 k_F x_{AB} \rangle \approx \frac{1}{2}$ for extended contacts

↳ $G_{nl} = 0$ ?

## d) higher order in tunneling




- 1d model $\quad H_0 = \frac{p^2}{2m} + U_0\,\delta(x)$

wave function $\quad \psi \sim \begin{cases} e^{ipx} + r e^{-ipx} & x<0 \\ t e^{ipx} & x>0 \end{cases} \qquad D = |t|^2$ transmission coefficient

$$\begin{cases} 1 + r = t \\ -t\frac{ip}{2m} + (1-r)\frac{ip}{2m} + U_0 t = 0 \end{cases} \Rightarrow \quad (1-t)\frac{ip}{m} + U_0 t = 0 \qquad t = \frac{1}{1 + i\frac{mU_0}{p}}$$

$$\Rightarrow \quad D = \frac{1}{1+Z^2} \qquad Z = \frac{mU_0}{p}$$

- $H_{BdG} = \begin{pmatrix} H_0 & \Delta \\ \Delta^* & -H_0 \end{pmatrix} \qquad \Delta(x) = \Delta\,\theta(x)$

~~$\psi(x) = \binom{1}{0} e^{\pm ipx}$ with energy $\varepsilon = \frac{p^2}{2m} - \mu$~~

~~$\psi(x) = \binom{0}{1} e^{\pm ipx}$ with energy $\varepsilon = -\frac{p^2}{2m} - \mu$~~

~~$\psi(x) = \binom{u}{v} e^{\pm ipx}$ with energy $\varepsilon = \sqrt{\left(\frac{p^2}{2m}-\mu\right)^2 + \Delta^2}$~~

We are looking for a scattering state with energy $\varepsilon < \Delta$ $\to$ it does not propagate in S (evanescent)

$\boxed{x>0}$

$$\psi(x) \sim \binom{u}{v} e^{ipx} \qquad \begin{pmatrix} \xi & \Delta \\ \Delta & -\xi \end{pmatrix}\binom{u}{v} = \varepsilon \binom{u}{v} \qquad \text{with } \xi = \frac{p^2}{2m} - \mu = \pm v\,(p \mp p_F)$$

for p close to $\pm p_F$

$$\varepsilon = \xi^2 + \Delta^2$$

i.e. $\xi = \pm i\sqrt{\Delta^2 - \varepsilon^2} \quad \Rightarrow \quad p = \pm p_F + \frac{i\sqrt{\Delta^2-\varepsilon^2}}{v}$

$$\frac{v}{u} = \frac{\varepsilon - \xi}{\Delta} = \underbrace{\frac{\varepsilon \mp i\sqrt{\Delta^2-\varepsilon^2}}{\Delta}}_{\text{Andreev phase shift}}$$

decay over length $\xi(\varepsilon) = \frac{v}{\sqrt{\Delta^2 - \varepsilon^2}}$

$$x>0 \quad \psi(x) \sim \left[ A\binom{1}{-i} e^{ip_F x} + B\binom{1}{i} e^{-ip_F x}\right] e^{-\frac{x}{\xi}} \qquad \text{for } \varepsilon \ll \Delta$$

$\boxed{x<0}$

$$\psi(x) \sim \binom{1}{r_A} e^{ip_F x} + \binom{r}{0} e^{-ip_F x}$$

matching:

$$\begin{cases} 1 + r = A + B \\ r_A = -i(A-B) \\ -\frac{ip_F}{2m}(A-B) + \frac{ip_F}{2m}(1-r) + U_0(1+r) = 0 \\ -\frac{ip_F}{2m}(-i)(A+B) + \frac{ip_F}{2m} r_A + U_0 r_A = 0 \end{cases}$$

$$\begin{cases} r_A + i(1-r) + 2Z(1+r) = 0 \\ -(1+r) + i r_A + 2Z r_A = 0 \end{cases} \longrightarrow \quad r = (2Z+i)\, r_A - 1$$

$$r_A = -(2Z+i) - (2Z-i).\left((2Z+i)\, r_A - 1\right)$$

$$r_A = \frac{-2i}{2 + 4Z^2}$$

$$= \frac{-i}{1 + 2Z^2} \qquad\qquad Z^2 = \frac{1}{D} - 1 = \frac{1-D}{D}$$

$$= \frac{-iD}{2-D}$$

$$R_A = \left(\frac{D}{2-D}\right)^2 \approx \begin{cases} \frac{D^2}{4} & D \to 0 \\ 1 & D \to 1 \end{cases}$$

$$\boxed{\begin{aligned} G_N &= \frac{2e^2}{\pi} \times D \\ G_A &= \frac{2e^2}{\pi} R_A \end{aligned}} \qquad \text{with} \qquad D = 4\pi^2 t_0^2 \nu_L \nu_R$$

[NB: higher order in tunnel Hamiltonian:
$$D = \frac{4\alpha}{(1+\alpha)^2} \qquad \alpha = \pi^2 t^2 \nu_L \nu_R$$
]

[$G_A \ll G_N$ in tunnel regime $D \ll 1$
$G_A = 2 G_N$ in ballistic regime $D = 1$]

<u>application</u>: perfect ballistic contact between $S$ and a ferromagnet

$$\begin{cases} G_N = \frac{e^2}{2\pi}(N_m + N_M) \\ G_A = \frac{2e^2}{\pi} N_m \end{cases}$$

$N_m$ and $N_M$: number of channels for minority and majority species

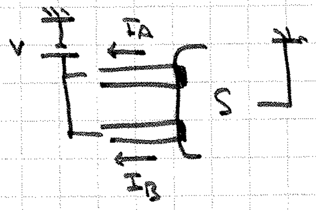
$$P = \frac{N_M - N_m}{N_M + N_m} = 1 - 2\,\frac{G_A/2}{2G_N} = 1 - \frac{G_A}{2G_N}$$

e) <u>noise</u>

- $I = \dfrac{\langle Q \rangle}{\tau}$     $Q$ charge transferred during a (long) time $\tau$

$$S = \int dt' \langle\langle I(t+t')\, I(t) \rangle\rangle = \frac{1}{\tau}\int_0^\tau dt \int_0^\tau dt' \langle\langle I(t+t')\, I(t)\rangle\rangle = \frac{\langle\langle Q^2 \rangle\rangle}{\tau}$$

with $\langle\langle Q^2 \rangle\rangle = \langle (Q - \langle Q \rangle)^2 \rangle$

- 

Pauli statistics and electron incoming 1 by 1:

| probability | $Q_A$ | $Q_B$ | $Q_A^2$ | $Q_A Q_B$ |
|---|---|---|---|---|
| $R_A$ | 2 | 0 | 4 | 0 |
| $R_A$ | 0 | 2 | 0 | 0 |
| $P_{CAR}$ | 1 | 1 | 1 | 1 |
| $1-2R_A-P_{CAR}$ | 0 | 0 | 0 | 0 |

$$\begin{cases} \langle Q_A \rangle = \langle Q_B \rangle = 2R_A + P_{CAR} \approx 2R_A & \text{si } P_{CAR} \ll R_A \\ \langle Q_A^2 \rangle \approx 4R_A \\ \langle Q_A Q_B \rangle = P_{CAR} \end{cases}$$

$$\begin{cases} I = 2R_A \times \frac{1}{\tau} \times e \\ S_{AA} = 4R_A(1-R_A)\,\frac{1}{\tau} \times e^2 \\ S_{AB} = (P_{CAR} - 4R_A^2) \times \frac{1}{\tau} \times e^2 \end{cases} \qquad \frac{1}{\tau} = \frac{eV}{h}$$

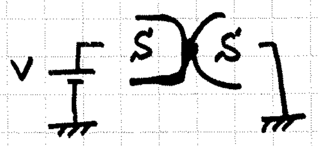
i) $\dfrac{S}{I} = \underbrace{2e}(1-R_A)$    vs    $\dfrac{S}{I} = \underbrace{e}(1-T)$ in N state

effective charge is doubled.

ii) $\begin{cases} S_{AB} < 0 & \text{in N state} \\ S_{AB} \text{ can become } > 0 & \text{in S state} \end{cases}$

# Lecture 3 : Andreev bound states

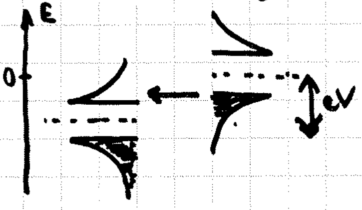
## a- Quasiparticle current [redacted]




### i) Tunnel junction

Within ~~Second~~ order perturbation theory (or Fermi Golden Rule), we find:

$$I(V) = \frac{G_N}{e}\int_{\substack{|E|>\Delta \\ |E+eV|>\Delta}} dE \, \frac{|E|}{\sqrt{E^2-\Delta^2}} \frac{|E+eV|}{\sqrt{(E+eV)^2-\Delta^2}} \left[ f(E) - f(E+eV) \right]$$




- at $T=0$, no current if $|eV| < 2\Delta$
- current near threshold: $eV = 2\Delta + \underbrace{(eV-2\Delta)}_{\text{small}}$ [redacted]

conditions $\left[\begin{array}{l} E<-\Delta \\ E+eV>\Delta \end{array}\right.$

$E \approx -\Delta + \underbrace{E+\Delta}_{\text{small}}$

[redacted] $\varepsilon$

[redacted]

$E+eV = \Delta +$ [redacted] $\varepsilon + eV - 2\Delta > \Delta$

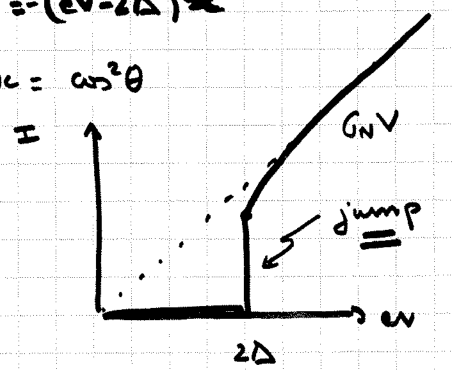
$$I(V) \approx \frac{G_N}{e} \int_{-(eV-2\Delta)}^{0} d\varepsilon \sqrt{\frac{\Delta}{-2\varepsilon}} \sqrt{\frac{\Delta}{2(\varepsilon + eV - 2\Delta)}}$$

$$= \frac{G_N \Delta}{2e} \int_0^1 dx \frac{1}{\sqrt{x(1-x)}}$$

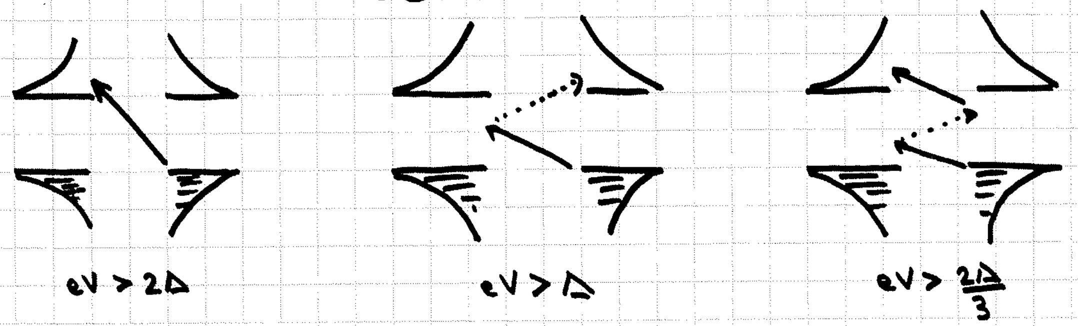
$$= \frac{\pi}{4} \frac{G_N \Delta}{e}$$

$\varepsilon = -(eV-2\Delta)x$

$x = \cos^2\theta$



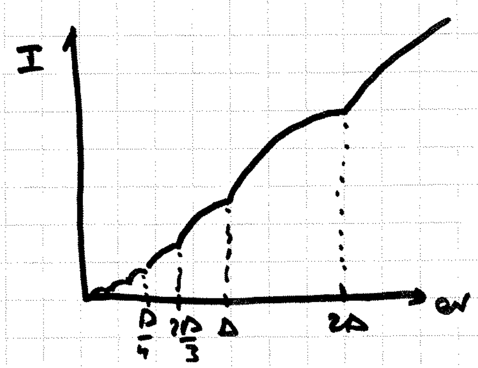

### ii) Multiple Andreev reflections




a new conduction channel opens when $eV_n > \frac{2\Delta}{n}$    $n$ integer

↪ multiparticle tunneling process

subgap features in the IV characteristics



## b. Andreev bound states in tunnel [redacted] junctions and quantum point contacts

In equilibrium, there are no propagating bound states below $\Delta$

However, there may be bound states localized near the junction.

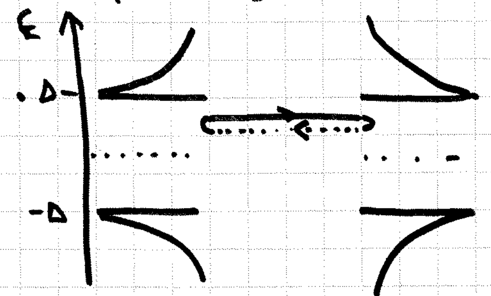
### i) balistic case

- according to the Bogoliubov transformation for a superconductor with phase $\varphi$

we found $$\frac{v(E)}{u(E)} = e^{i\varphi}\sqrt{\frac{E-\xi}{E+\xi}} \qquad \text{with } \xi = \sqrt{E^2-\Delta^2} = \pm i\sqrt{\Delta^2-E^2}$$
if $|E|<\Delta$

i.e. [redacted] the phase [redacted]

acquired by [redacted] Andreev reflected hole is $\varphi \mp \arccos\left(\frac{E}{\Delta}\right)$




There is a bound state if

$$\varphi \pm 2\arccos\frac{E}{\Delta} = 2n\pi$$

$$\boxed{E = \pm\Delta\cos\frac{\varphi}{2}}$$

$\varphi$ is the superconducting phase difference.

### ii) tunnel case :

- $H = H_L + H_R + H_T$ $\qquad \varphi = \varphi_L - \varphi_R$

$$= \sum_{k\sigma} E_k \alpha^+_{k\sigma}\alpha_{k\sigma} + \sum_{p\sigma} E_p \beta^+_{p\sigma}\beta_{p\sigma} + t_0\sum_{kp\sigma} e^{-i\varphi}\left(u_k\alpha^+_{k\sigma} - \sigma v_k\alpha_{-k-\sigma}\right)\left(u_p\beta_{p\sigma} - \sigma v_p\beta^+_{-p-\sigma}\right) + h.c.$$

for $\xi_k \to 0$ $\qquad E_k \simeq \Delta + \frac{\xi_k^2}{2\Delta}$ $\qquad u_k, v_k \approx \frac{1}{\sqrt{2}}$

We project $H$ on the subspace of states with energy close to the continuum threshold $\Delta$

↳ an excited state contains at most one excitation.

$$H_{eff} \approx \sum_{k\sigma}\left(\Delta + \frac{\xi_k^2}{2\Delta}\right)\alpha^+_{k\sigma}\alpha_{k\sigma} + \sum_{p\sigma}\left(\Delta + \frac{\xi_p^2}{2\Delta}\right)\beta^+_{p\sigma}\beta_{p\sigma}$$ [redacted] $$i t_0 \sin\varphi \sum_{kp\sigma}\left(\alpha^+_{k\sigma}\beta_{p\sigma} - h.c.\right)$$

let us introduce $$\gamma_{k\sigma} = \frac{1}{\sqrt{2}}\left(\alpha_{k\sigma} + i\beta_{k\sigma}\right)$$ (which preserve fermionic commutation relations)
$$\delta_{k\sigma} = \frac{1}{\sqrt{2}}\left(\alpha_{k\sigma} - i\beta_{k\sigma}\right)$$

$$\alpha = \frac{1}{\sqrt{2}}(\gamma+\delta)$$
$$\beta = -\frac{i}{\sqrt{2}}(\gamma-\delta)$$
$$\alpha^+\beta - \beta^+\alpha$$ [redacted]
$$= -\frac{i}{2}\left[(\gamma^+ + \delta^+)(\gamma-\delta) + (\gamma^+ - \delta^+)(\gamma+\delta)\right]$$

Then,

$$H_{eff} = \sum_{k\sigma}\left(\Delta + \frac{\xi_k^2}{2\Delta}\right)\left(\gamma^+_{k\sigma}\gamma_{k\sigma} + \delta^+_{k\sigma}\delta_{k\sigma}\right)$$
$$+ t_0\sin\varphi\sum_{kk'\sigma}\left(-\gamma^+_{k\sigma}\gamma_{k'\sigma} + \delta^+_{k\sigma}\delta_{k'\sigma}\right)$$

This Hamiltonian is identical to that of a massive particle in 1D in the presence of a potential well or barrier

$\downarrow$ in that case, it always admits for a bound state!

$$H_{eff} = \Delta + \frac{p^2}{2m} \pm U\delta(x)$$

To obtain the bound state energy, we look for a trial wavefunction:

$$|\psi\rangle = \sum_k C_k \gamma^{+}_{k\uparrow} |BCS\rangle \qquad H|\psi\rangle = \epsilon|\psi\rangle$$

$$\left(\epsilon - \Delta - \frac{\xi_k^2}{2\Delta}\right) C_k + \sum_q t_0 \sin\frac{\varphi}{2} C_q = 0$$

$$\hookrightarrow \quad C_k = -\frac{t_0 \sin\frac{\varphi}{2}}{\epsilon - \Delta - \frac{\xi_k^2}{2\Delta}} \sum_q C_q$$

$$\hookrightarrow \quad 1 = + t_0 \sin\frac{\varphi}{2} \sum_k \frac{1}{\frac{\xi_k^2}{2\Delta} + \Delta - \epsilon} =$$

$$= \nu t_0 \times 2\Delta \times \sin\frac{\varphi}{2} \int d\xi \, \frac{1}{\xi^2 + 2\Delta(\Delta - \epsilon)}$$

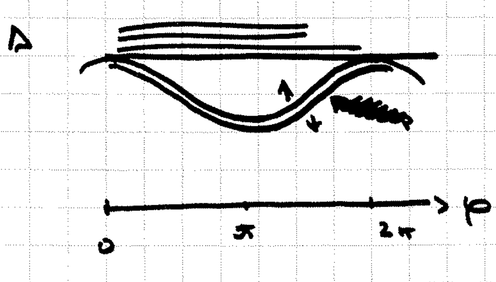
$$= \pi \nu t_0 \sqrt{\frac{2\Delta}{\Delta - \epsilon}} \sin\frac{\varphi}{2}$$

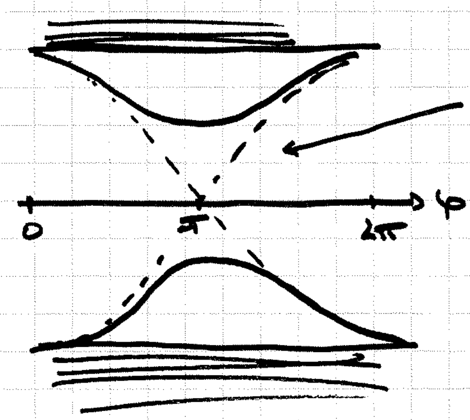
$\longrightarrow$ bound state with energy $\quad \epsilon = \Delta\left(1 - 2(\pi\nu_0 t_0)^2 \sin^2\frac{\varphi}{2}\right)$

if $\; 0 < \varphi < \pi$.

- the bound state is spin degenerate
- the bound state is in the sector $\delta$ if $\pi < \varphi < 2\pi$ $\longrightarrow$ one bound state merges with the continuum and another one appears.
- normal state transmission $D = (2\pi\nu_0 t_0)^2 \longrightarrow \epsilon_A = \Delta\left(1 - \frac{D}{2}\sin^2\frac{\varphi}{2}\right)$

The general formula is $\quad \epsilon_A = \Delta\sqrt{1 - D\sin^2\frac{\varphi}{2}}$




at $D = 1$, the bound state crosses the Fermi level

at $D < 1$, there is an avoided level crossing with a gap $E_g = 2\Delta\sqrt{R}$

- The phase dependent part of the junction energy is

$$E_J(\varphi) = E_A(\varphi)\sum_\sigma\left(\langle \gamma^+_{A\sigma}\gamma_{A\sigma}\rangle - \frac{1}{2}\right)$$

$$= -E_A(\varphi)\,\mathrm{th}\left(\frac{E_A(\varphi)}{2T}\right) + \text{cste}$$

$\hookrightarrow$ in the tunneling case $\quad E_A(\varphi) = \Delta - \frac{D\Delta}{2}\sin^2\frac{\varphi}{2}$, $\quad \sin^2\frac{\varphi}{2} = \frac{1-\cos\varphi}{2}$

$$E_J(\varphi) \approx -E_J\,\mathrm{th}\frac{\Delta}{2T}\cos\varphi \qquad \text{with } E_J = \frac{D\Delta}{4} = \frac{\pi}{4}G_N\Delta$$

iii) consequences :

- Let us perform a gauge transformation on the BCS Hamiltonian with $U = e^{i\frac{\varphi(t)}{2}\sum_{k\sigma}a^+_{k\sigma}a_{k\sigma}}$

$H_{eff} = U^+ H_{BCS} U - i\hbar U^+\dot{U}$ is the equivalent Hamiltonian.

$$= \sum_{k\sigma}\Big(\xi_k - \underbrace{\hbar\frac{\dot\varphi}{2}}_{eV}\Big)a^+_{k\sigma}a_{k\sigma} + \Delta e^{i\varphi(t)}\sum_k a^+_{k\uparrow}a^+_{-k\downarrow} + \text{h.c.}$$

i.e. : an electric potential acting on the electrons should come with a time-dependent superconducting phase, with $\boxed{\dot\varphi = \frac{2eV(t)}{\hbar}}$

- Equilibrium : the free energy variation of the junction can be related with the electric work:
$$\delta E_J(\varphi) = I \times V \times \delta t = I \times \frac{\hbar}{2e}\delta\varphi$$
Thus $\quad I_J = \frac{2e}{\hbar}\frac{\partial E_J}{\partial\varphi} \quad$ equilibrium supercurrent

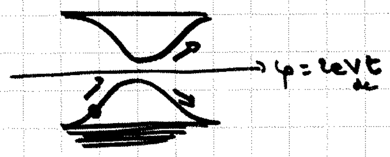
for the QPC: $$I = \frac{\pi}{2}\frac{G_N\Delta}{e}\frac{\sin\varphi}{\sqrt{1-D\sin^2\frac{\varphi}{2}}}\,\mathrm{th}\left(\frac{\Delta}{2T}\sqrt{1-D\sin^2\frac{\varphi}{2}}\right)$$

* tunnel limit: $I \approx I_c\sin\varphi \qquad I_c = \frac{\pi}{2}\frac{G_N\Delta}{e}$ at $T=0$
(Ambegaokar - Baratoff relation)
* the Josephson relation may contain higher harmonics: $I = \sum_{n=1}^{\infty} I_n\sin(n\varphi)$
(no $\cos(n\varphi)$ if time reversal symmetry : $I(-\varphi) = -I(\varphi)$)
* the current depends on the ABS occupations:

current $\quad I \quad -I \quad \underbrace{0 \quad 0}_{\text{poisonned states}}$

"even sector" $\qquad$ "odd sector" $\rightarrow$ charges differs by e [20]

• **Non equilibrium** $V = V_{dc}$ (more in lecture 4)




near the avoided crossing $D \to 1$

$$H_{eff} \approx \begin{pmatrix} \Delta eVt & \Delta\sqrt{R} \\ \Delta\sqrt{R} & -\Delta eVt \end{pmatrix}$$

a Landau-Zener process can switch the ABS occupation:

It releases an energy $2\Delta$ during a period $T = \frac{2\pi}{2eV}$, with a probability $P_{LZ} = e^{-\frac{(\Delta\sqrt{R})^2}{\Delta eV}} = e^{-\frac{R\Delta}{eV}}$

i.e. $IV = \frac{2\Delta}{T} P_{LZ} \rightarrow I(V) = \frac{2e\Delta}{\pi} e^{-\frac{R\Delta}{eV}}$

$\#_{MAR}$

**MAR and ABS are related !**

## c. ABS in quantum dots

We consider a quantum dot that contains a single, spin degenerate discrete level (or we assume a large level spacing to ignore the effect of other levels) with strong Coulomb repulsion $\rightarrow$ Anderson model

$$H = \sum_\sigma \varepsilon d^+_\sigma d_\sigma + U n_\uparrow n_\downarrow + H_S + H_T \qquad n_\sigma = d^+_\sigma d_\sigma$$

$$H_T = t \sum_{k\sigma} a^+_{k\sigma} d_\sigma + h.c.$$

energy scales: $\varepsilon$, $U$, $\Delta$, $\Gamma = 2\pi \nu_S |t|^2$
level broadening due to the coupling of the level to the lead (in N state)

### i) singlet/doublet transition

we assume $\Delta \gg \varepsilon, U, \Gamma$ so that we can derive an effective Hamiltonian:

$$H_{eff} = H_{dot} - H_T \frac{1}{H_S} H_T$$

$$= \varepsilon \sum_\sigma d^+_\sigma d_\sigma + U n_\uparrow n_\downarrow + \Gamma d^+_\uparrow d^+_\downarrow + h.c.$$

4 eigenstates: $|\uparrow\rangle = d^+_\uparrow|\emptyset\rangle$, $|\downarrow\rangle = d^+_\downarrow|\emptyset\rangle$ } with energy $\varepsilon$

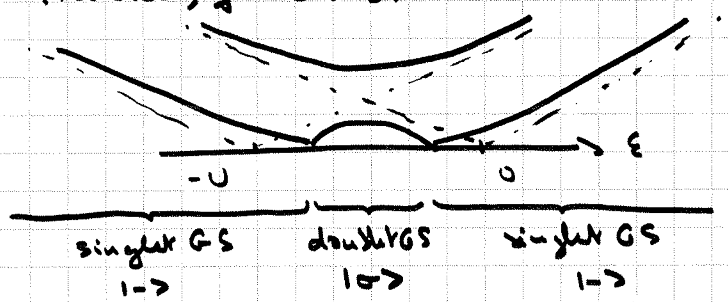
$|+\rangle = (v - u\, d^+_\uparrow d^+_\downarrow)|\emptyset\rangle$ with energy $E_+ = \varepsilon + \frac{U}{2} + \sqrt{\left(\varepsilon+\frac{U}{2}\right)^2 + \Gamma^2}$

$|-\rangle = (u + v\, d^+_\uparrow d^+_\downarrow)|\emptyset\rangle$ ——— $E_- = \varepsilon + \frac{U}{2} - \sqrt{(\;)^2 + \Gamma^2}$

↳ $|-\rangle$ and $|\sigma\rangle$ compete for being the ground state:

$|-\rangle$ is always the ground state if $U < 2\Gamma$

However, if $U > 2\Gamma$




## ib) Competition between Superconductivity and Kondo effect

At large $U$, a Kondo effect may occur in the normal state in the doublet region: How does it resist to superconducting correlations?

- Kondo Hamiltonian: $\varepsilon \approx -U/2$

$$H = \sum_{k\sigma} \xi_k c^+_{k\sigma} c_{k\sigma} + H_\Delta + H_{dot} + H_T$$

$$\approx H_N + H_\Delta - \sum_{ss'} |s\rangle\langle s| H_T \frac{1}{H_d} H_T |s'\rangle\langle s'| \qquad |s\rangle = d^+_s|\emptyset\rangle$$

$$-t^2 \sum_{\substack{ss' \\ k\sigma, k'\sigma'}} \Big( |s\rangle\langle s| c^+_{k\sigma} d_\sigma \frac{1}{U/2} d^+_{\sigma'} c_{k'\sigma'} |s'\rangle\langle s'| + |s\rangle\langle s| d^+_{\sigma'} c_{k'\sigma'} \frac{1}{U/2} c^+_{k\sigma} d_\sigma |s'\rangle\langle s'| \Big)$$

1st line: $\sigma' = \bar{s}'$ $\quad \sigma = \bar{s}$

2nd line: $\sigma = s'$ $\quad \sigma' = s$

$$-\frac{2t^2}{U} \sum_{k\sigma k'\sigma'} c^+_{k\sigma} c_{k'\sigma'} \left( \sigma\sigma' |\bar{\sigma}'\rangle\langle\bar{\sigma}| - |\sigma'\rangle\langle\sigma| \right)$$

$$\begin{pmatrix} \uparrow\uparrow & \downarrow\downarrow - \uparrow\uparrow \\ \downarrow\downarrow & \uparrow\uparrow - \downarrow\downarrow \\ \uparrow\downarrow & -2\downarrow\uparrow \\ \downarrow\uparrow & -2\uparrow\downarrow \end{pmatrix}$$

$$H = H_S + J \sum_{k\sigma k'\sigma'} \vec{S} \cdot c^+_{k\sigma} \vec{\sigma}_{\sigma\sigma'} c_{k'\sigma'} \qquad \begin{cases} \vec{S} = \frac{\hbar}{2}\vec{\sigma} \\ J = \frac{4t^2}{U} \end{cases}$$

{ AF coupling between a local spin (on the dot) and the spins in the conduction band.

- Classical spin: $\vec{S} = S\hat{z}$

⇝ Same approximation as for the tunnel junction:

$$H = \sum_{k\sigma}\left(\Delta + \frac{\xi_k^2}{2\Delta}\right)\alpha^+_{k\sigma}\alpha_{k\sigma} + JS\sum_{\sigma k k'}\sigma\left(u_k\alpha^+_{k\sigma} - \sigma\alpha_{k\bar{\sigma}}\right)\left(u_{k'}\alpha_{k'\sigma} - \sigma v_{k'}\alpha^+_{-k'\bar{\sigma}}\right)$$

$$\approx \quad \underline{\qquad\qquad\qquad\qquad} \quad + JS\sum_{\sigma}\sigma\,\alpha^+_{k\sigma}\alpha_{k'\sigma}$$

→ bound state with energy $E_S = \Delta\left(1 - 2(\pi\nu JS)^2\right)$

for $\sigma = \ominus$ only.

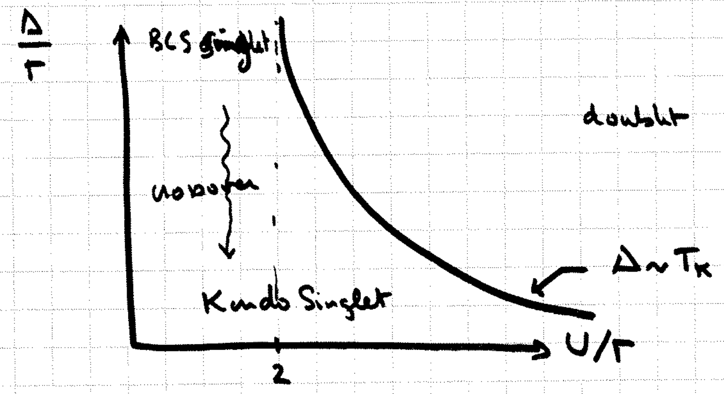
Shiba state

- Quantum spin: Kondo problem: $J \nearrow$ as one looks at smaller and smaller energy scales

$J_{eff} \sim \dfrac{1}{\ln\frac{\Delta}{T_K}}$ when $\Delta \gg T_K$ $\qquad T_K \sim \sqrt{U\Gamma}\,e^{-\frac{U}{\Gamma}}$

at $\Delta \sim T_K$ $\quad E_S \sim 0$ → transition from doublet to singlet (Kondo) ground state.

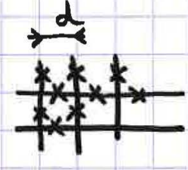
- Phase diagram:



**Lecture 4 : Josephson effects** Consequence of $I(\varphi) = I_c \sin\varphi$ (BCS ground state, no qp.)

## a. Classical Josephson effect :

### i) Meissner effect




- Superconductor as the continuum limit of a Josephson junction array
- We found previously $I = I_c \sin\left(\varphi_{\lambda c} + \frac{2e}{\hbar}\int^t ds\, V(s)\right)$ → depends on electromag. gauge

with $I_c = \frac{\pi}{2}\frac{G_N \Delta}{e}$ (Ambegaokar-Baratoff relation for tunnel junctions)

Let us apply it to a diffusive metal : $G_N = \frac{\sigma S}{d}$

The current density is $\vec{j} = \frac{I_c d}{S}\left(-\vec{\nabla}\varphi + \frac{2e}{\hbar}\int^t ds\, \vec{E}(s)\right)$ [after linearisation of the Josephson relation]

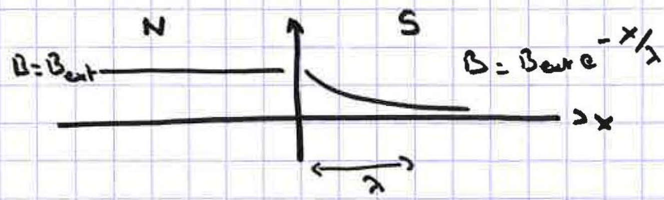
Which yields in another gauge : $\vec{j} = -\frac{\pi\sigma\Delta}{e}\underbrace{\left(\vec{\nabla}\varphi + \frac{2e}{\hbar c}\vec{A}\right)}$

Same combination as in the Schrödinger equation for a particle in field with charge $q = -2e$
⟷ expected from the GL equations.

London equation:

$$\boxed{\vec{j} = -\frac{c}{4\pi\lambda^2}\vec{A}}$$

- Consequence:



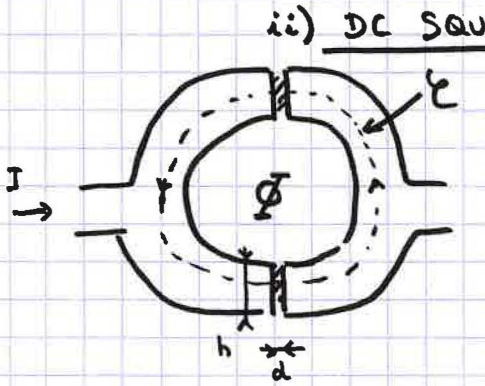

$$\begin{cases} \text{rot}\, B = \frac{4\pi}{c} j = -\frac{A}{\lambda^2} \\ B = \text{rot}\, A \\ \text{London Gauge : div}\, A = 0 \quad V = 0 \end{cases}$$

$\vec{B} = B(x)\,\hat{z} = \frac{dA}{dx}\hat{z}$

$\vec{A} = A(x)\,\hat{y}$

$\frac{d^2A}{dx^2} = \frac{1}{\lambda^2}A$ ➡ [ Screening of the external magnetic field by the supercurrent

### ii) DC SQUID:




we assume
- $\lambda \gg h \gg d$ : the field is screened along the contour $\mathcal{C}$ : $j_s = 0$
- $\lambda d B \ll \Phi_0$ : each JJ is not affected by the field that wants to penetrate the insulating layer (we will understand the criterion soon)

$\Phi_0 = \frac{hc}{2e} = \frac{\pi c}{e}$ is the flux quantum.

N.B. the trapped flux is quantized in units of $\Phi_0$ in a thick ring

Then, $I = I_c \sin\varphi_{up} + I_c \sin\varphi_{down}$ ~~up~~

$\varphi_{up} = \varphi^L_{up} - \varphi^R_{up}$ $\qquad$ $\varphi_{down} = \varphi^L_{down} - \varphi^R_{down}$

$$0 = \int_{\mathcal{C}_1} d\vec{l}\cdot\left(\vec{\nabla}\varphi + \frac{2e}{\hbar c}\vec{A}\right) = \varphi^L_{down} - \varphi^L_{up} + \frac{2e}{\hbar c}\int_{\mathcal{C}_1} d\vec{l}\cdot\vec{A}$$

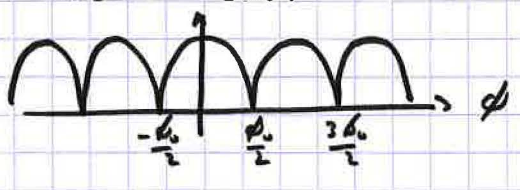
$$0 = \int_{\mathcal{C}_2} d\vec{l}\cdot\left(\vec{\nabla}\varphi + \frac{2e}{\hbar c}\vec{A}\right) = \varphi^R_{up} - \varphi^R_{down} + \frac{2e}{\hbar c}\int_{\mathcal{C}_2} d\vec{l}\cdot\vec{A}$$

$$\Rightarrow \oint_{\mathcal{C}} d\vec{l}\cdot\vec{A} = BS = \Phi = -\frac{\hbar c}{2e}\left[\varphi^R_{up} - \varphi^R_{down} - \varphi^L_{up} + \varphi^L_{down}\right]$$

$$= \frac{\hbar c}{2e}\left[\varphi_{down} - \varphi_{up}\right] = -\frac{\Phi_0}{2\pi}\left[\varphi_{down} - \varphi_{up}\right]$$

i.e. $I = I_c\left[\sin\varphi_{up} + \sin\left(\varphi_{up} - \frac{2\pi\Phi}{\Phi_0}\right)\right]$

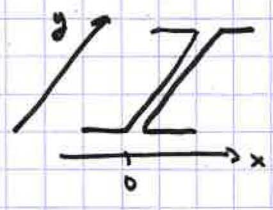
$$\boxed{I_{max} = 2I_c\left|\cos\frac{\pi\Phi}{\Phi_0}\right|}$$




→ Accurate magnetometer

→ $I_{max}$ decreases when a superconducting flux is added

- Similar effect along a single extended junction:




$\vec{B} = B e^{-|x|/\lambda}\,\hat{z}$ $\qquad$ $\lambda \gg d$

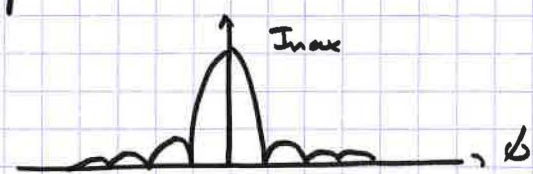
$\vec{A} \approx -B(x)\,y\,\hat{x}$

$$j_x = j_c \sin\left(\varphi - \frac{2e}{\hbar c}\int dx\, A(x)\right) \approx j_c \sin\left(\varphi - \frac{2\pi}{\Phi_0}\times 2\lambda B y\right)$$

$$I = j_c\int_0^h dy\, \sin\left(\varphi - \frac{2\pi}{\Phi_0}\times 2\lambda B y\right)$$

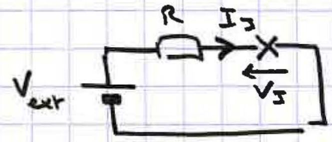
$$I_{max} = I_c\left(\frac{\sin \pi\Phi/\Phi_0}{\pi\Phi/\Phi_0}\right) \qquad \Phi = 2\lambda . h . B$$

Fraunhofer pattern




$\frac{\Phi}{\Phi_0}$ is the number of flux quanta in the junction

iii) AC Josephson effect :



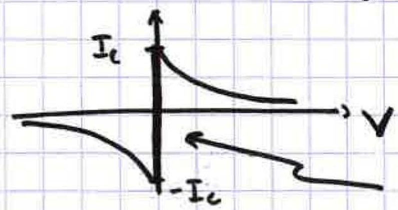

$$V_{ext} = R I_J + V_J$$
$$= R I_c \sin\varphi + \frac{\hbar}{2e}\dot\varphi$$

if $V_{ext} < R I_c$, stationary solution $\varphi = \arcsin\left(\frac{V_{ext}}{R I_c}\right) \Rightarrow \left(\begin{array}{l} I_J < I_c \\ V_J = 0 \end{array}\right.$

if $V_{ext} > R I_c$, periodic solution with $\dot\varphi \neq 0$

$\varphi(t+T) = \varphi(t) + 2\pi \Rightarrow V = \frac{\hbar}{2e}\frac{2\pi}{T}$

$$\int_0^{2\pi} \frac{d\varphi}{V_{ext} - R I_c \sin\varphi} = \frac{2e}{\hbar} T = \frac{2\pi}{V}$$
$$\shortparallel$$
$$\frac{2\pi}{\sqrt{V_{ext}^2 - R I_c^2}} \longrightarrow \begin{array}{l} V_J = \sqrt{V_{ext}^2 - R I_c^2} \\ I_J = \frac{1}{R}(V_{ext} - V) \end{array}$$

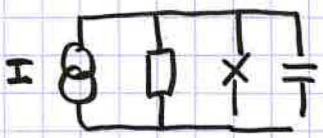



NB: $R I_c \ll \frac{\Delta}{e}$

and $I_c \sim \frac{G_T \Delta}{e}$

$\Rightarrow$ the model is OK if $R \ll R_T$

- In the litterature, we frequently see:




RS(C)J : Resitively Shunted (Capacitance and) Josephson junction

↳ without capacitance, it directly transforms into the equations already discussed.

$$I = I_c \sin\varphi + C\frac{\hbar}{2e}\ddot\varphi + \frac{\hbar}{2eR}\dot\varphi$$

- anharmonic oscillator
- underdamped vs overdamped

- If $V \gg R I_c$ (and $I \to 0$)

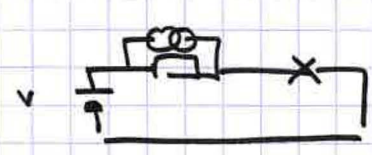
then $\varphi \simeq \varphi_0 + 2eVt$

$I(t) = I_c \sin(2eVt + \varphi_0)$

↳ it depends on an initial phase ?? (unphysical)

We may introduce a simple model for fluctuations :

thermal noise due to the resistance:




Langevin source that reproduce the fluctuation dissipation relation:

$\langle \delta i(t) \rangle = 0$

$\langle \delta i(t)\delta i(t') \rangle = \frac{2T}{R}\delta(t)$

Random Gaussian white noise

As a result, the supercurrent branch would get rounded (not discarded) and:

$$\langle I(t)\rangle = \langle I_c \sin\left(2eVt + \varphi_0 + \int_0^t ds\, 2eR\, \delta i(s)\right)\rangle$$
$$= \mathrm{Im}\, I_c\, e^{i(\varphi_0+2eVt)} \langle e^{2ieR\int_0^t ds\, \delta i(s)}\rangle$$
$$= \mathrm{Im}\, I_c\, e^{i(\varphi_0+2eVt)}\, e^{-\frac{1}{2}(2eR)^2\int^t ds\int^t ds' \langle \delta i(s)\delta i(s')\rangle}$$
$$= I_c \sin(\varphi_0 + 2eVt) \times e^{-4e^2RT|t|} \xrightarrow[t \gg \frac{1}{4e^2RT}]{} 0$$

$$S(\omega) = \int d\tau\, e^{-i\omega\tau}\langle I(t)I(t+\tau)\rangle$$
$$= \frac{I_c^2}{4}\oint d\tau\, e^{i(2eV-\omega)\tau}\langle e^{2ieR\int_t^{t+\tau} ds\, \delta i(s)}\rangle$$ [redacted].
$$= \frac{I_c^2}{4}\int d\tau\, e^{i(2eV-\omega)\tau}\, e^{-4e^2RT|\tau|}$$
$$= \frac{\pi I_c^2}{2}\;\underbrace{\frac{1}{\pi}\frac{\Gamma}{(\omega-2eV)^2+\Gamma^2}}_{\delta_\Gamma(\omega-2eV)}$$

Josephson radiation at frequency $\omega = 2eV$ with a width $\Gamma \sim \frac{R}{R_Q} T$

- It is more practical to apply dc+ac bias:

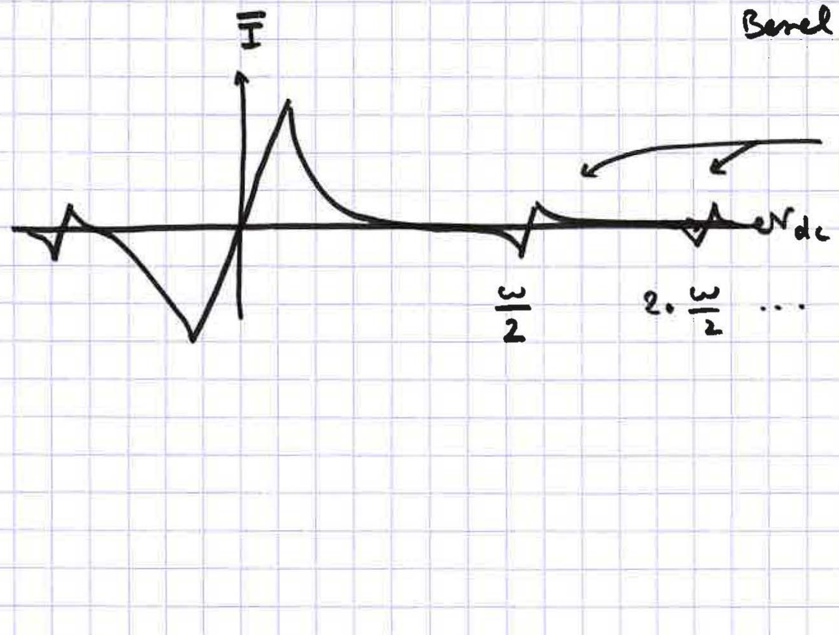
$$e^{ix\cos\omega t} = \sum_n J_n(x) e^{in\omega t}$$

$$V_{dc} + V_{ac}\cos\omega t = R I_c \sin\varphi + \frac{\hbar}{2e}\dot{\varphi} \quad (1)$$

with $\varphi =$ [redacted] $n\omega t + \frac{2eV_{ac}}{\omega}\sin\omega t + \chi$

(1) becomes: $V_{dc} - \frac{\hbar}{2e} n\omega = R I_c \sin\left(\chi +\right.$ [redacted] $\left. n\omega t + \frac{2eV_{ac}}{\omega}\sin\omega t\right) + \frac{2e}{\hbar}\dot{\chi}$

$$\approx R I_c \underbrace{J_{-n}\left(\frac{2eV_{ac}}{\omega}\right)}_{\text{Bessel function}} \sin\chi + \frac{2e}{\hbar}\dot{\chi}$$




replicas of the supercurrent branch at $2eV_{dc} = n\omega$ ($n$ integer)

= Shapiro steps

## b) Quantum Josephson effect

### i) macroscopic quantum tunneling

- Let us reconsider the RCSJ model:

$$I = I_c \sin\varphi + C\frac{\hbar}{2e}\ddot{\varphi} + \underbrace{\frac{\hbar}{2eR}\dot{\varphi}}_{\text{friction term neglected for a while}} \qquad I_c = \frac{2e}{\hbar}E_J \qquad \varphi = \frac{2e\Phi}{\hbar} \qquad \dot{\Phi} = V$$

$$H = \frac{Q^2}{2C} - E_J\cos\left(\frac{2e}{\hbar}\Phi\right) - I\Phi$$

Classical equations of motion:

$$\dot{Q} = \frac{\partial H}{\partial \Phi} = I_c \sin\Phi - I$$ is the current through the capacitance

$$\dot{\Phi} = -\frac{\partial H}{\partial Q} = -\frac{Q}{C}$$ → reproduces the RCSJ equation at $R \to \infty$

$$H = \frac{Q^2}{2C} + \underbrace{U(\varphi)}_{\text{washboard potential}}$$


$I < I_c$ : stationary solution

$I > I_c$ : the phase runs out $\langle\dot{\varphi}\rangle \neq 0$

$C$ ~ inverse mass : it determines the inertia for the phase.

$C = 0$ : mass is infinite → RSJ model non hysteretic

$C \neq 0$ : possible hysteresis.

If the particle is blocked in a well,


there is a finite probability to escape.

- At finite temperature: escape rate given by an Arrhenius law

$$\Gamma \sim e^{-\frac{\Delta U}{T}} \qquad \Delta U = f(I - I_c)$$

- the rate saturates at low $T$: quantum tunneling !

- Quantization of the phase: $[Q, \Phi] = i\hbar$ $\qquad Q = -2en$

$$\boxed{[\varphi, n] = i}$$

* Uncertainty principle $\Delta\varphi\,\Delta n \sim 1$ : Either the phase or the charge is fixed (already seen in the case of a grain)
* $[\varphi, n] = i$ ⟷ quantisation of the electromagnetic field in the circuit.

## ii) Cooper pair box

$V_g$

$$H = -E_J \cos\varphi + E_c (n - \mathcal{N}_g)^2$$

- $E_J \ll E_c$: change basis $\quad [\varphi, n] = i \longrightarrow [e^{\pm i\varphi}, n] = \mp e^{\pm i\varphi}$

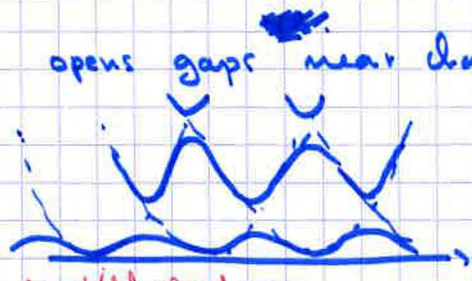
i.e. $H = \sum_n E_c (n - \mathcal{N}_g)^2 |n\rangle\langle n| - \frac{E_J}{2} \underbrace{\left(|n\rangle\langle n+1| + |n+1\rangle\langle n|\right)}_{\text{coherent transfer of Cooper pairs.}}$

$E_J \to 0 \qquad \psi_n(\varphi) \approx e^{in\varphi}$

$\psi_n(\varphi + 2\pi) = \psi_n(\varphi)$ describes the same state $\Rightarrow$ $n$ is an integer.

(i.e. the phase is defined in a ring $[0, 2\pi]$)

$E_J$ small opens gaps near charge degeneracy points $-(n - \mathcal{N}_g) = n + \frac{1}{2} - \mathcal{N}_g$



$\updownarrow \frac{E_J^2}{E_c}$ ($2^{nd}$ order P.T.) i.e. $\mathcal{N}_g = n + \frac{1}{2}$

$\updownarrow E_J$ ($1^{st}$ order P.T.)

$\mathcal{N}_g$

localized in N / delocalized in $\varphi$

$\rightarrow$ two level system: qubit!

- $E_c \ll E_J$: phase basis $\quad n = -i \frac{\partial}{\partial\varphi} \qquad \psi(\varphi) = e^{i\mathcal{N}_g\varphi} u(\varphi)$

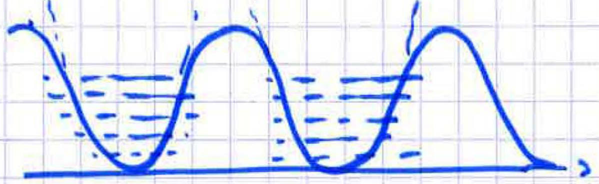
absorbs $\mathcal{N}_g$

Then, the Josephson term want to impose $\varphi \simeq 2\pi n + \underset{\text{small}}{\delta\varphi}$

i.e. $H \simeq \frac{E_J}{2} \varphi^2 - E_c \frac{\partial^2}{\partial\varphi^2}$

is the equation of an harmonic oscillator

energy levels $\quad \omega_n = \omega_p (n + \frac{1}{2}) \qquad \omega_p = \sqrt{2 E_J E_c}$ is a plasma frequency



$\varphi$

localized in $\varphi$ / delocalized in N

The wave functions behave like $\qquad \psi_n(\varphi) \sim e^{-\sqrt{\frac{E_J}{E_c}} (\varphi - 2\pi n)^2}$

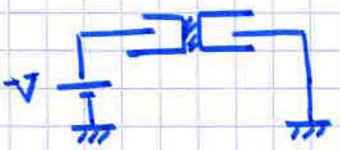
Exponentially small overlap $\rightsquigarrow$ formation of bands of width $\# e^{-\# \sqrt{\frac{E_J}{E_c}}}$

$E \psi_m = E_m \psi_m + t (\psi_{m+1} + \psi_{m-1})$

$E_k = E_m + 2t \cos(2\pi k)$

$\rightarrow$ not sensitive to $\mathcal{N}_g$: protection from charge noise!

iii) Bloch oscillations




same equations but now, the phase is not restricted to $[0,2\pi]$

$H_0 = -E_J\cos\varphi + E_c n^2$

Bloch theorem: $\psi_k(\varphi) = e^{ik\varphi} u_k(\varphi)$ $\quad |k| < \frac{1}{2}$

$u_k(\varphi+2\pi) = u_k(\varphi)$

Same calculation: $E_0(k) = E_0 - 2t\cos(2\pi k)$

at $E_J \gg E_c$

$H = H_0 - I\frac{\hbar}{2e}\varphi \quad \rightsquigarrow \quad \dot{k} = -\frac{\partial H}{\partial\varphi} = \frac{\hbar}{2e} I$

$\dot{\varphi} = +\frac{\partial H}{\partial k} = +\Delta\sin(2\pi k)$

$\Rightarrow$ for $I = cst$ $\quad k = \frac{\hbar}{2e} I \times t + cst$

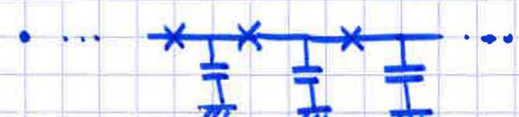
$\langle V\rangle \sim \Delta\sin\left(\frac{\pi I t}{e} + cst\right)$

$\Downarrow$

Bloch frequency $\omega_B = \frac{I}{2e}$ — not observed fully convincingly

dual of the Josephson frequency $\omega_J = \frac{2eV}{\hbar}$

iv) Superconductor/Insulator transition:



$H = \sum_i E_c n_i^2 - E_J\cos(\varphi_i - \varphi_j)$

At large $E_J$, one can linearize the Josephson relation $\to$ transmission line with a tunable impedance $\quad Z \sim \frac{h}{e^2}\sqrt{\frac{E_c}{E_J}} \sim \sqrt{\frac{L_{eff}}{C}} \sim \sqrt{\frac{\mu_0}{\varepsilon_0}} \sim \mu_0 c \quad C = \varepsilon_0 l$, $L = \mu_0 l$

while with waveguides: $Z_{vac} \sim \sqrt{\frac{L_{geom}}{C}} \sim \frac{Z_Q}{\alpha}$ $\quad \alpha \sim$ fine structure constant.

- mean field: $H = E_c n^2 - \underbrace{E_J\langle\cos\varphi\rangle}_{E_{J\,eff}}\cos\varphi$

with $\quad \langle\cos\varphi\rangle = -\left\langle\frac{\partial H}{\partial E_{J\,eff}}\right\rangle$ $\quad$ Self consistency equation.

i.e. $\quad E_{J\,eff} = -E_J\frac{\partial E_g}{\partial E_{J\,eff}} \qquad E_g \sim -\frac{E_{J\,eff}^2}{E_c}$ at $E_{J\,eff} \to 0$

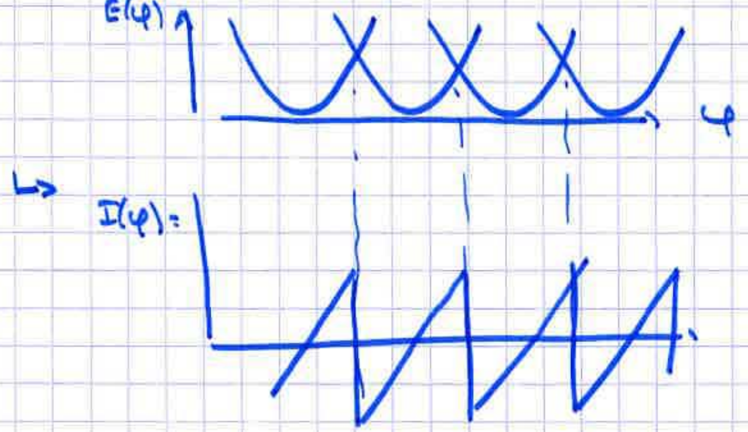
$\boxed{1 \sim E_J/E_c}$ critical value for the transition.

## v) Quantum phase slips :

model for a nanowire ..  ...

phase bias $\varphi$: each junction of a chain of $N$ elements sees a phase $\frac{\varphi}{N}$

i.e. $E(\varphi) = N\, E_J \left(\frac{\varphi - 2\pi m}{N}\right)^2 = \frac{E_J}{N} (\varphi - 2\pi m)^2$




$I(\varphi)$ has a sawtooth behavior.

hopping from one well to another is a phase slip event tuned by capacitive coupling ⇝ rounding of the current phase characteristics,

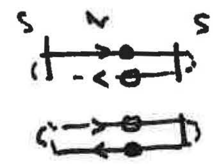
- × more complicated arrays.
- × vortices Aharonov Bohm & Aharonov Casher effect
- × BKT

# Lecture 5 : Proximity effect

How do superconducting correlations leak into a non-superconducting metal?

## 1. N/S hybrids

### a) ballistic regime




- Bohr-Sommerfeld quantization in 1d geometry

$$-2\arccos\frac{\varepsilon}{\Delta} \pm \frac{2\varepsilon L}{v_F} \mp \varphi = 2n\pi$$

phase at AR — phase accumulated along the N part

the spectrum is particle-hole symmetric and $2\pi$-periodic

if $\frac{v_F}{L} \ll \Delta$, the subgap spectrum is given by

$$\varepsilon_{n\pm} = \frac{\pi v_F}{L}\left(n + \frac{1}{2} \pm \frac{\varphi}{2\pi}\right)$$

$$H = \sum_i \varepsilon_i\left(\gamma_i^\dagger \gamma_i - \frac{1}{2}\right)$$

- The total energy is

$$E = -\frac{1}{2}\int d\varepsilon\, \varepsilon\, \text{th}\frac{\varepsilon}{2T} \underbrace{\sum_{n\pm} \delta(\varepsilon - \varepsilon_{n\pm})}_{\sum_{k\pm} \frac{L}{\pi v_F} \underbrace{e^{-2i\pi k\left(\frac{\varepsilon L}{\pi v_F} - \frac{1}{2} \mp \frac{\varphi}{2\pi}\right)}}_{(-)^k e^{\pm ik\varphi} e^{-2i\frac{\varepsilon L}{v_F}k}}}$$

Poisson summation formula

$$I = 2e\frac{\partial E}{\partial \varphi} = \sum_{k\pm} \mp \frac{ieL}{\pi v_F} k(-)^k e^{\pm ik\varphi} \underbrace{\int d\varepsilon\, \varepsilon\, \text{th}\frac{\varepsilon}{2T} e^{-2i\frac{\varepsilon L}{v_F}k}}_{\frac{iv_F}{2L}\frac{\partial}{\partial k}\underbrace{\int d\varepsilon\, \text{th}\frac{\varepsilon}{2T} e^{-2i\frac{\varepsilon L k}{v_F}}}}$$

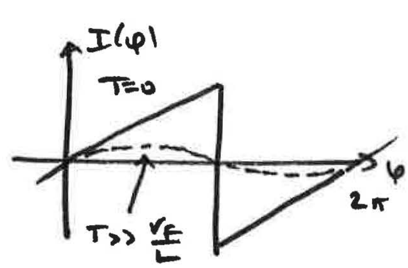
$$= \int d\varepsilon\, 2T \sum_{i\omega} \frac{1}{\varepsilon - i\omega} e^{-2i\frac{\varepsilon L k}{v_F}} = -\frac{2i\pi T}{\text{sh}\left(\frac{\pi T\, 2Lk}{v_F}\right)}$$

$$= \frac{2ev_F}{\pi L}\sum_{k=1}^{\infty} \frac{(-)^{k+1}}{k}\sin k\varphi\, \frac{x_k^2\, \text{ch}\, x_k}{\text{sh}^2 x_k} \qquad x_k = \frac{2\pi T L}{v_F}k$$

$T=0$: $\quad I(\varphi) = \frac{ev_F}{\pi L}\varphi \qquad |\varphi| < 2\pi$

Sawtooth behavior

$T \gg \frac{v_F}{L}$: $\quad I(\varphi) \simeq 16\pi \frac{eT^2 L}{v_F} e^{-\frac{2\pi T L}{v_F}} \sin\varphi$



→ $\frac{v_F}{L}$ sets the energy scale for the minigap and critical current at $T=0$

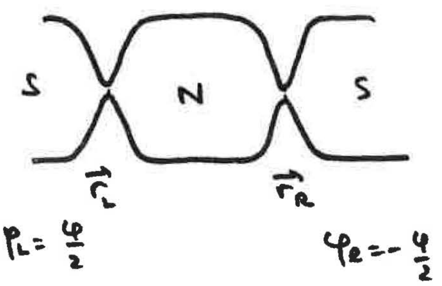
→ The critical current does not vanish at $T > \frac{v_F}{L}$

i.e. proximity effect extends on a long scale $\frac{v_F}{T}$

→ similar to persistent current in normal ring pierced with a flux, but many channels contribute (can ...)

b) diffusive regime

• S/I/N/I/S geometry




Tunnel point contact:

$$G_T = \frac{2e^2}{h} D \qquad D \ll 1$$

$$D = 4\pi^2 \nu_S \nu_N t^2$$

$\Delta$ largest energy scale: effective Hamiltonian for the transfer of Cooper pairs into Andreev pairs

$$H \simeq H_N + H_{TL}^{eff} + H_{TR}^{eff}$$

$$H_N = \sum_{k\sigma} \xi_k c^\dagger_{k\sigma} c_{k\sigma}$$

$$H_{T\alpha}^{eff} = -\pi \nu_S t^2 e^{i\varphi_\alpha} \psi^\dagger_\uparrow(r_\alpha) \psi^\dagger_\downarrow(r_\alpha) + h.c.$$

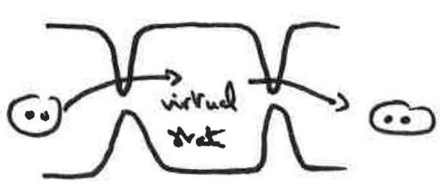
• 2nd order perturbation theory for the ground state energy:

$$E_J(\varphi) = - \langle H_T \frac{1}{H_N} H_T \rangle$$

$$= - e^{-i\varphi} (\pi \nu_S t^2)^2 \Big[ \langle \psi^\dagger_\uparrow(r_R) \psi^\dagger_\downarrow(r_R) \frac{1}{H_N} \psi_\downarrow(r_L) \psi_\uparrow(r_L) \rangle + \langle \psi_\downarrow(r_L) \psi_\uparrow(r_L) \frac{1}{H_N} \psi^\dagger_\uparrow(r_R) \psi^\dagger_\downarrow(r_R) \rangle \Big]$$

$+ h.c.$

$+$ terms independent of $\varphi$



- We introduce exact eigenstate in N: $\quad H_N \chi_\ell = \xi_\ell \chi_\ell$

$$\psi_\sigma(r) = \sum_k \chi_\ell(r)\, a_{\ell\sigma}$$

$\chi_\ell(r)$ wavefunction associated with energy $\xi_\ell$ (not necessary plane wave!)

$$\text{Then } E_J(\varphi) = -e^{-i\varphi}(\pi\nu_s t^2)^2 \sum_{\ell m}\Big[\chi^*_\ell(r_R)\chi^*_m(r_R)\,\frac{f(\xi_\ell)f(\xi_m)}{-\xi_\ell-\xi_m}\,\chi_m(r_R)\chi_\ell(r_R)$$
$$+ \;''\; \frac{[1-f(\xi_\ell)][1-f(\xi_m)]}{\xi_\ell+\xi_m}\;''\;\Big]$$

$$+\dots$$

$$= -2\cos\varphi\,(\pi\nu_s t^2)^2 \int d\xi\, d\xi'\, \frac{1-f(\xi)-f(\xi')}{\xi+\xi'}\, K_\xi(r_L,r_R) K_{\xi'}(r_L,r_R)$$

+ terms independent of $\varphi$,

where $K_\xi(r_L r_R) = \sum_\ell \chi^*_\ell(r_R)\chi_\ell(r_R)\,\delta(\xi-\xi_\ell)$ spectral function

$G^{R/A}_\xi(r_L-r_R) = \sum_\ell \dfrac{\chi^*_\ell(r_R)\,\chi_\ell(r_L)}{\xi-\xi_\ell\pm i0^+}$ retarded and advanced Green function

$$\left(\begin{array}{l} \hookrightarrow\; E = -E_J\cos\varphi + \text{cste} \\ I = I_c \sin\varphi \\ I_c = 2e E_J \end{array}\right) \text{ sinusoidal current-phase relation.}$$

$$\boxed{K = -\frac{1}{2i\pi}\left(G^R - G^A\right)}$$

$\downarrow$ Green functions in disordered metals.

- Disordered metal

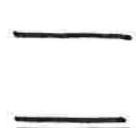
$$H = \frac{p^2}{2m} + U(r)$$

$$\langle U(r)\rangle = 0$$

$$\langle U(r)U(r')\rangle = U_0^2\,\delta(r-r') \quad \text{white noise potential.}$$

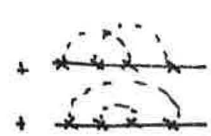
Green function average over the disorder:

$$G_0^{R/A} = \frac{1}{\varepsilon - H_0 \pm i0^+}$$



terms neglected if $k_F\ell \gg 1$

$$\langle G^{R/A}\rangle = \frac{1}{\varepsilon - H_0 - \Sigma}$$



$\xi_0 \qquad \xi_0$

$$\langle G\rangle = G_0 + G_0\,\Sigma\, G$$

$$\Sigma^{R/A} = U_0^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\varepsilon - \xi_p - \Sigma^{R/A}} \qquad \text{ansatz: } \Sigma^{R/A} = \mp \frac{i}{2\tau}$$

$$= U_0^2 \int \frac{d^3p}{(2\pi)^3} \left( \mathcal{P} \frac{1}{\varepsilon - \xi} \mp i\pi\, \delta(\varepsilon - \xi_p) \right)$$

$= \mp i\pi\nu U_0^2$ + real part which can be absorbed by redefining the chemical potential

i.e $\boxed{\frac{1}{\tau} = 2\pi\nu U_0^2}$ scattering rate for ~~coll~~ elastic collisions on impurities.

• Then $G^{R/A}(r) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}.\vec{r}}}{\varepsilon - \xi_p \pm \frac{i}{2\tau}}$ $\qquad \vec{p}.\vec{r} = r\left(p_F + \frac{\xi}{v_F}\right)\cos\theta$

for $\xi \ll p_F v_F \sim \varepsilon_F$ (large energy)

$$= \frac{\nu}{2ip_F r} \int d\xi \, \frac{e^{i\left(p_F + \frac{\xi}{v_F}\right)r} - e^{-i\left(p_F + \frac{\xi}{v_F}\right)r}}{\varepsilon - \xi \pm \frac{i}{2\tau}}$$

$$= \frac{i\nu}{2p_F r} \times (\pm 2i\pi) \times (\pm)\, e^{\pm i\left(p_F + \frac{\varepsilon}{v_F} \pm \frac{i}{2\tau}\right)r}$$

$$= -\frac{\pi\nu}{p_F r} e^{\pm i p_F r} e^{-\frac{r}{2l}} \qquad l = v_F\tau$$

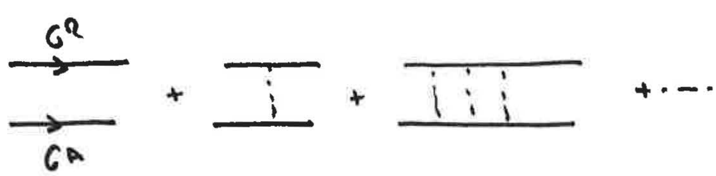
$$\langle K(r) \rangle = \nu \frac{\sin p_F r}{p_F r} e^{-r/2l}$$

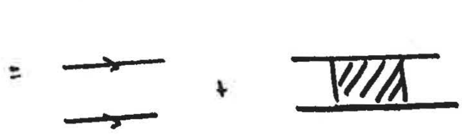
$K(0) = \nu$ is the density of states !

$K(r) \longrightarrow 0$ for $r \gg l$

• However $\langle K_\xi(r_1 r_2) K_{\xi'}(r_1 r_2) \rangle$ decays on a much longer scale.

$$= \frac{1}{4\pi^2} \langle G^R_\xi(r_1 r_2) G^A_{\xi'}(r_1 r_2) \rangle + (\xi \leftrightarrow \xi')$$





with

$$\Pi_{\xi-\xi'}(q) = U_0^2 + U_0^2 \underbrace{\int \frac{d^3p}{(2\pi)^3} \langle G^R_\xi(p)\rangle\langle G^A_{\xi'}(p-q)\rangle}_{I} \Pi_{\xi-\xi'}(q)$$

$$\Pi_{\xi-\xi'}(q) = \frac{U_0^2}{1 - U_0^2 I}$$

↳ why one needs R & A !

$$I = \nu \int d\xi \int \frac{d\Omega}{4\pi} \frac{1}{\xi - \xi_p + \frac{i}{2\tau}} \; \frac{1}{\xi - \xi_p - \bar{v}.\bar{q} - \frac{i}{2\tau}}$$

$$= 2i\pi\nu \int \frac{d\Omega}{4\pi} \frac{1}{\frac{i}{\tau} + \xi - \xi' - v.q} \qquad |\xi - \xi'|, |v.q| \ll \frac{1}{\tau} \ll \varepsilon_F$$

$$\simeq 2\pi\nu\tau \left(1 + i\tau(\xi - \xi') - \tau D q^2\right) \qquad D = \frac{v^2\tau}{3}$$

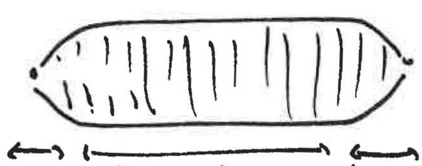
is the diffusion constant

i.e. $$\Pi_{\xi-\xi'}(q) = \frac{1}{2\pi\nu\tau^2} \overbrace{\frac{1}{Dq^2 - i(\xi-\xi')}}^{D_{\xi-\xi'}(q)}$$

↳ diffuson

Typical scale $L \sim \sqrt{\frac{D}{k_B T}} \gg l$

• Now:




$$\langle K_\xi(r_L r_R) K_{\xi'}(r_L r_R)\rangle = \frac{1}{4\pi^2} \frac{1}{\nu^2} \int dr dr' \overbrace{G^R_\xi(r_L r)}^{G^A_\xi(r_L r)} \overbrace{G^A_{\xi'}(r' r_R)}^{G^A(r' r)} \Pi_{\xi-\xi'}(r r') + h.c.$$

$$\approx \frac{1}{4\pi^2} \times (2\pi\nu\tau)^2 \times 2 \,\mathrm{Re}\, \Pi_{\xi-\xi'}(q)$$

$$\hookrightarrow I_c = \underbrace{4e \times (\pi\nu_d t^2)^2 \times (\nu\tau)^2 \frac{1}{2\pi\nu\tau^2}}_{\frac{e D^2 \delta}{8\pi^3}} \times \int d\xi d\xi' \frac{1 - f(\xi) - f(\xi')}{\xi + \xi'} 2 \,\mathrm{Re}\, D_{\xi-\xi'}(r_L r_R)$$

$$\xi = \varepsilon + \frac{\omega}{2}$$
$$\xi' = \varepsilon - \frac{\omega}{2}$$

$$I_c = \frac{eD^2\delta}{8\pi^3}\sum_k e^{ik(r_L-r_R)}\int\frac{d\varepsilon\, d\omega}{2\varepsilon}\, 2T\sum_{i\omega_n}\left(\frac{1}{\varepsilon+\frac{\omega}{2}+i\omega_n}+\frac{1}{\varepsilon-\frac{\omega}{2}-i\omega_n}\right)\left(\frac{1}{Dk^2-i\omega}+\frac{1}{Dk^2+i\omega}\right)$$

$\downarrow$
allows making the integration over $\varepsilon$ !

non. of $\omega$s

$$= \frac{eD^2\delta}{8\pi^3}\sum_k e^{ik(r_L-r_R)}\; 2i\pi T\sum_{\omega_n>0}\left(\frac{1}{2i\omega_n+\omega}-\frac{1}{-2i\omega_n+\omega}\right)\left(\frac{1}{Dk^2-i\omega}+\frac{1}{Dk^2+i\omega}\right)$$

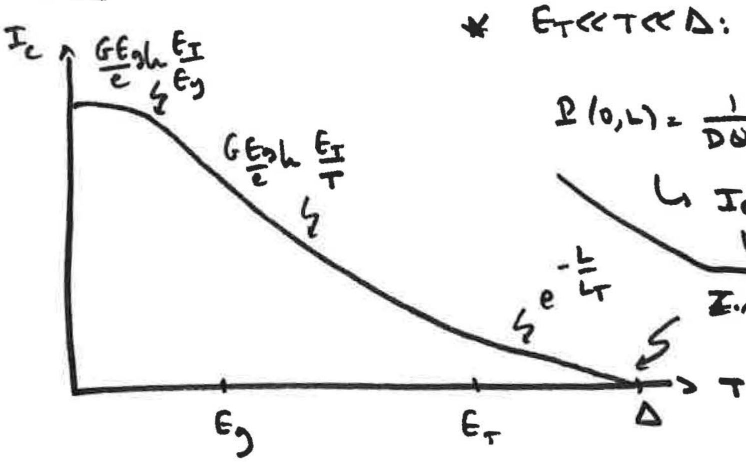
$$= \frac{eD^2\delta}{\pi}\, T\sum_{\omega_\nu\geq 0}\int\sum_k\frac{e^{ik(r_L-r_R)}}{Dk^2+2\omega_\nu}$$

* $T \ll E_T \ll \Delta$ :   $I_c \sim \dfrac{eD^2\delta}{4\pi^2}\ln\dfrac{E_T}{T}$

"zero mode approximation

$eGE_g$
(Ambegaokar Baratoff like relation with $\Delta \to E_g$)

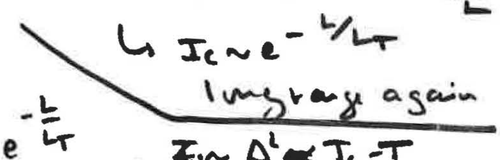
divergence cut off by $E_g \sim D\delta \sim \dfrac{G}{G_Q}\delta$




* $E_T \ll T \ll \Delta$:  $(-D\nabla^2 + 2\omega_\nu)P(x,x') = \delta(x-x')$
$\nabla P(0,x) = \nabla P(x,L) = 0$

$P(0,L) = \dfrac{1}{DQ\,\mathrm{sh}\,QL}$   $Q = \sqrt{\dfrac{2\omega}{D}}$

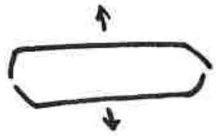
$\hookrightarrow I_c \sim e^{-L/L_T}$




- many energy scales
- the proximity effect is non perturbative
- Non linear equation for the "Cooper" $\Pi_{\xi,\xi'}(G)$ = Usadel equation.
  $\hookrightarrow$ minigap.

## 2 - F/S hybrids:

### a) Oscillating proximity effect:

• 

$\xi \to \xi + h$
$\xi' \to \xi' - h$

$\xi - \xi' \to \xi - \xi' + 2h$

NB: also works in the [redacted] ballistic case:
$I_{SFS}(\varphi) = \frac{1}{2}\sum_{\pm} I_{SNS}\left(\varphi \pm \frac{hL}{\pi v_F}\right)$

$$I_c \sim \mathrm{Re}\, T \sum_{k,\omega} \frac{e^{ik(r_L - r_R)}}{Dk^2 + 2(\omega_\nu + ih)}$$

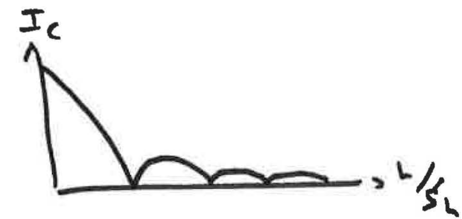
$$\sim \mathrm{Re} \sqrt{\frac{D}{2ih}}\, e^{-L\sqrt{\frac{2ih}{D}}}$$

$$\sim \mathrm{Re}\, (1+i)\, e^{-\frac{L}{\xi_h}(1+i)} \qquad \xi_h = \sqrt{\frac{2h}{D}}$$

$$\sim e^{-L/\xi_h} \sin\left(\frac{L}{\xi_h} + \frac{\pi}{4}\right)$$

decay — oscillations

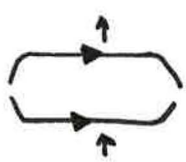
$\pi$-junction behavior.




as $h \gg T_c$ typically
$\xi_h \ll \sqrt{D/T}$

$\Rightarrow$ the proximity effect is short-ranged.

### b) Long range triplet proximity effect

• 

would be insensitive to the exchange field in F
$\to$ long range triplet proximity effect?

a local virtual Hamiltonian does not work, even for a magnetic barrier
(Pauli principle $\psi_\uparrow(r)\psi_\uparrow(r) = 0$)
but it can ~~be~~ occur for an extended barrier.

• [redacted]da of nota Canada

# IV - Topological superconductivity

## 1 - Majorana fermions

- Majorana found real solutions of the Dirac equation (1937)
  ↳ fermion which is its own antiparticle : $\gamma = \gamma^+$
- A Dirac fermion can always be viewed as a combination of two Majorana fermions:

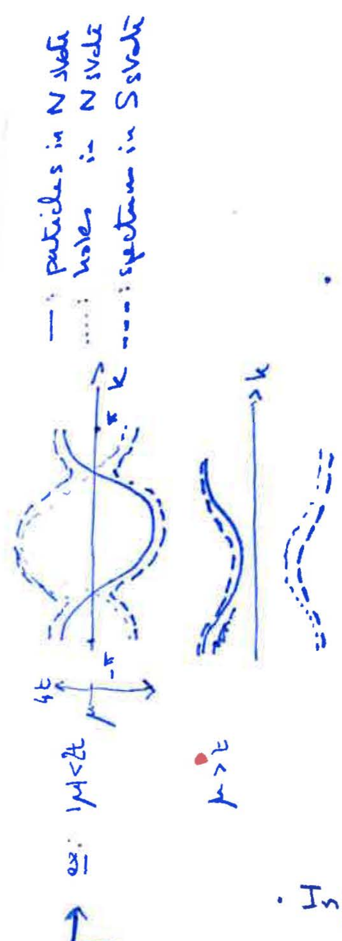
$$\begin{cases} c = \frac{1}{\sqrt{2}}(\gamma_1 + i\gamma_2) \\ c^+ = \frac{1}{\sqrt{2}}(\gamma_1 - i\gamma_2) \end{cases} \longleftrightarrow \begin{cases} \gamma_1 = \frac{1}{\sqrt{2}}(c + c^+) \\ \gamma_2 = -\frac{i}{\sqrt{2}}(c - c^+) \end{cases}$$

with $\gamma_1^2 = \gamma_2^2 = \dfrac{1}{2}$ and $c^+c = i\gamma_1\gamma_2 + \frac{1}{2}$

- no fundamental particle is known to be a Majorana fermion: neutrino?
- emergent low energy excitations in condensed matter physics:

  FQHE $\nu = 5/2$
  B phase of Helium 3 (superfluid)
  → various models involving superconductivity

## 2 - The Kitaev model (2001)




- In the BCS approach of Sec I.1-, we found $\gamma_\uparrow(E) = \gamma_\downarrow^+(-E)$
  i.e. look for Majorana fermions in a spinless model:

$$H = \sum_n -t(c_n^+ c_{n+1} + c_{n+1}^+ c_n) - \mu c_n^+ c_n + \Delta(c_n^+ c_{n+1}^+ + c_{n+1} c_n)$$

1d tight binding model on N sites

$t$ : hopping
$\mu$ : chemical potential
$\Delta$ : pair potential that cannot act on-site (Pauli principle)
→ $\Delta(k) \propto \sin k_x$ "p-wave" symmetry for the Cooper wave function to be symmetric

- Infinite chain: $H = \sum_k (c_k^+ \ c_{-k}) \begin{pmatrix} -2t\cos k - \mu & \Delta \sin k \\ \Delta \sin k & 2t\cos k + \mu \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^+ \end{pmatrix}$

the spectrum is $E_k = \pm\sqrt{(2t\cos k + \mu)^2 + \Delta^2 \sin^2 k}$. It is always gapped except when $|\mu| = 2t$ → What happens when the gap closes and reopens (as $\mu$ varies)?

We now introduce Majorana fermions such that

$$c_n = \frac{1}{\sqrt{2}}\left(\Gamma_{An} + i\Gamma_{Bn}\right)$$
$$c_n^+ = \frac{1}{\sqrt{2}}\left(\Gamma_{An} - i\Gamma_{Bn}\right)$$

$$\hookrightarrow H = -i\sum_n \left\{ (t+\Delta)\Gamma_{An}\Gamma_{Bn+1} - (t-\Delta)\Gamma_{Bn}\Gamma_{An+1} \right\} \mu \sum_n \left( i\Gamma_{An}\Gamma_{Bn} + \right.$$

• specific cases:

• $t = \Delta = 0$ $\qquad H = -\mu \sum_i c_i^+ c_i$

$\hookrightarrow$ unique ground state (vaccum)
excitations cost energy $|\mu|$ $\quad \Big\} \leftarrow$ trivial state

• $t = \Delta \qquad \mu = 0 \qquad H = -it \sum_n \Gamma_{An}\Gamma_{Bn+1} = t\sum_n d_n^+ d_n$

where $d_n = \frac{1}{\sqrt{2}}\left(\Gamma_{Bn+1} + i\Gamma_{An}\right)$

$\hookrightarrow$ Majorana mode on neighbouring sites are coupled
excitations from vacuum cost energy $|t|$

**but** ~~these~~ $\Gamma_{AN}$ and $\Gamma_{B1}$ do not appear in the Hamiltonian

$\Rightarrow$ a highly non local fermion with zero energy is formed out of these edge states. $\quad \Big\} \leftarrow$ topologically non trivial state.

• topological transition:

$|\mu| < 2t$ : $\quad$ 1 pair of Majorana end states (not localized on a single site in general)

$|\mu| > 2t$ : $\quad$ no zero energy state

NB: The zero energy state corresponds to a different parity of the total number of electrons whether it is filled or empty (up to charge 2e absorbed by the condensate)

NB: topological index $n \in \mathbb{Z}_2 = \{0,1\}$

$n$ = # crossings of the Fermi surface in N state (at $k>0$)
= # pairs of end states

# 3- Experimental realizations

- Helical edges of a quantum spin Hall insulator (cf. L. Lévy's lecture)

$$H = \int dx \Big[ -i v_F \left[ \psi_R^\dagger \partial_x \psi_R - \psi_L^\dagger \partial_x \psi_L \right] - \mu \left( \psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L \right)$$
$$+ h \left[ \psi_R^\dagger \psi_L + \psi_R^\dagger \psi_L \right) + \Delta \left( \psi_R^\dagger \psi_L^\dagger + \psi_L \psi_R \right) \Big]$$

R-movers with spin ↑
L-movers with spin ↓
$h$ = transverse field
$\Delta$ = effective pairing induced by proximity with a superconductor

ex: HgTe/CdTe or InAs/GaSb heterostructures

Introducing Pauli matrices $\begin{cases} \sigma_i & \text{in R/L space} \\ \tau_i & \text{in particle/hole space} \end{cases}$ we obtain the Bogoliubov Hamiltonian

$$H = (v_F k \sigma_3 - \mu)\tau_z + h\sigma_x + \Delta \tau_x$$

$\hookrightarrow H^2 = v_F^2 k^2 + \mu^2 + h^2 + \Delta^2 - 2 v_F k \mu \sigma_z - 2 h\mu \sigma_x \tau_z + 2 h \Delta \sigma_x \tau_x$

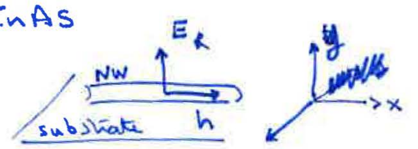
$\hookrightarrow \left[ H^2 - (v_F^2 k^2 + \mu^2 + h^2 + \Delta^2) \right]^2 = 4 \left[ (v_F k \mu)^2 + (h\mu)^2 + (h\Delta)^2 \right]$

i.e. spectrum $\quad E_k = \pm \sqrt{v^2 k^2 + \mu^2 + h^2 + \Delta^2 \,(\pm)' 2 \sqrt{v^2 k^2 \mu^2 + h^2(\mu^2 + \Delta^2)}}$

with a gap at $k = 0$ $\quad 2 \left| h - \sqrt{\mu^2 + \Delta^2} \right|$

→ trivial insulator with a "spin" gap if $h > \sqrt{\mu^2 + \Delta^2}$
↳ topological superconductor if $h < \sqrt{\mu^2 + \Delta^2}$
( p wave superconductor with $\Delta(k) = \Delta \,\mathrm{sgn}(k)$)

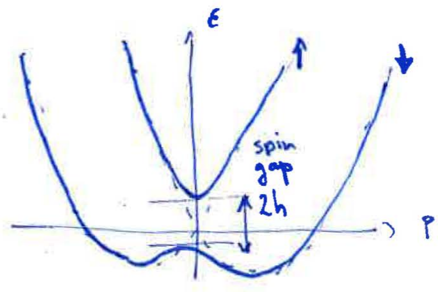
- Semiconducting nanowires ex: InSb, InAs




+

normal state : $\quad H = \dfrac{p^2}{2m} + \underbrace{\alpha p \sigma_z}_{\text{Rashba term}} + h\sigma_x$

( $h \parallel E_R$ also works)




$$= \frac{1}{2m}\left( p + m\alpha \sigma_z \right)^2 + h\sigma_x + \text{cste}$$

now with induced superconductivity, the Bogoliubov Hamiltonian reads

$$H = \frac{1}{2}\sum_p \Psi_p^+ \left[ \left(\frac{p^2}{2m} - \mu + \alpha p \sigma_z\right)\tau_z + \Delta \tau_x + h\sigma_x \right] \Psi_p$$

with $\Psi_p = \begin{pmatrix} c_{p\uparrow} \\ c_{p\downarrow} \\ c^+_{-p\downarrow} \\ -c^+_{-p\uparrow} \end{pmatrix}$

the spectrum in superconducting state again displays a gap

$$E_g = 2\left| h - \sqrt{\Delta^2+\mu^2} \right|$$

but now, $\begin{cases} h < \sqrt{\Delta^2+\mu^2} & \text{trivial superconductor} \\ h > \sqrt{\Delta^2+\mu^2} & \text{topological superconductor} \end{cases}$

<u>NB</u>: ~~[illegible]~~ effective p-wave superconductivity:

We first diagonalize the normal part of the spectrum:

$$H_N = \sum_{p\sigma} a^+_{p\sigma} \left( \frac{p^2}{2m} - \mu \pm \sqrt{h^2 + (\alpha p)^2} \right) a_{p\sigma}$$

$\sigma = \uparrow/\downarrow$ correspondent aux deux bands de la page 22

with $\begin{pmatrix} a_{p\uparrow} \\ a_{p\downarrow} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} c_{p\uparrow} \\ c_{p\downarrow} \end{pmatrix}$ avec $\cos\frac{\theta}{2} = \frac{\alpha p}{\sqrt{h^2+(\alpha p)^2}}$

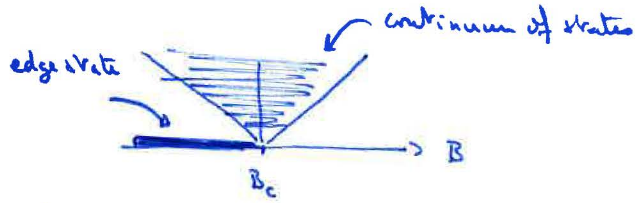
When $\alpha p_F \sim m\alpha^2$, $\mu \ll h$, there is effectively only one band ($\downarrow$) at the Fermi level. Projecting the Hamiltonian on that band, we obtain

$$H \approx \sum_p \Big\{ \varepsilon_p a^+_{p\downarrow} a_{p\downarrow} - \underbrace{\Delta \frac{\alpha p}{h}}_{\text{p-wave pairing}} a^+_{p\downarrow} a^+_{-p\downarrow} + h.c. \Big\}$$

# 4- Signatures of Majorana fermions

**Zero-bias anomaly:**

- typical spectrum




by performing a tunneling spectroscopy at the edge of a superconductor one should observe a ZBA on the topological side

- Low energy Hamiltonian (energies << superconducting gap)

$$H = \underbrace{\sum_k \xi_k c_k^+ c_k}_{\text{normal lead}} + \underbrace{t\,\gamma \sum_k (c_k - c_k^+)}_{\text{point contact}} \qquad (1)$$

$\gamma = c + c^+$


An incident electron at energy $\varepsilon$ may be either reflected as an electron or as a hole

electron → no transferred current

hole → Andreev reflection process carrying a charge 2e

We rewrite in real space: (Bogoliubov de Gennes approach)

$$H = \frac{1}{2}\int dx \Big\{ (\psi^+(x) \;\; \psi(x)) \begin{pmatrix} -iv\partial_x & 0 \\ 0 & +iv\partial_x \end{pmatrix} \begin{pmatrix} \psi(x) \\ \psi^+(x) \end{pmatrix}$$
$$+ (c^+ \;\; c) \begin{pmatrix} t & -t \\ t & -t \end{pmatrix} \delta(x) \begin{pmatrix} \psi(x) \\ \psi^+(x) \end{pmatrix}$$
$$+ (\psi^+(x) \;\; \psi(x)) \begin{pmatrix} +t & +t \\ -t & -t \end{pmatrix} \delta(x) \begin{pmatrix} c \\ c^+ \end{pmatrix} \Big\}$$

This Hamiltonian can be diagonalized with the Bogoliubov transformation

$$\psi(x) = \sum_\varepsilon \left(u_\varepsilon(x)\gamma_\varepsilon + v_\varepsilon^*(x)\gamma_\varepsilon^+\right)$$
$$c = \sum_\varepsilon \left(\alpha\,\gamma_\varepsilon + \beta^*\gamma_\varepsilon^+\right)$$

provided that

$$\begin{cases} \varepsilon\, u_\varepsilon(x) = -iv_F\partial_x u_\varepsilon(x) + t\,\delta(x)(\alpha+\beta) \\ \varepsilon\, v_\varepsilon(x) = +iv_F\partial_x v_\varepsilon(x) - t\,\delta(x)(\alpha+\beta) \\ \varepsilon\,\alpha = t[u(0) - v(0)] \\ \varepsilon\,\beta = +t[u(0) - v(0)] \end{cases}$$

We rewrite in real space (similar to Bogoliubov-de Gennes transformation):

$$H = \frac{1}{2}\int dx \Big\{ \left(\psi^+(x)\;\psi(-x)\right)\begin{pmatrix} -iv_F\partial_x & 0 \\ 0 & iv_F\partial_x \end{pmatrix}\begin{pmatrix}\psi(x)\\ \psi^+(-x)\end{pmatrix}$$
$$+ \overbrace{\left(c^+\;\; c\right)}^{\delta(x)}\begin{pmatrix} t & -t \\ t & -t\end{pmatrix}\begin{pmatrix}\psi(x)\\ \psi^+(-x)\end{pmatrix}$$
$$+ \overbrace{\left(\psi^+(x)\;\psi(-x)\right)}^{\delta(x)}\begin{pmatrix} t & +t \\ -t & -t\end{pmatrix}\begin{pmatrix}c\\ c^+\end{pmatrix}\Big\}$$

We introduce the transformation

$$\psi(x) = \sum_{\varepsilon>0} u_\varepsilon(x)\,\gamma_\varepsilon + v^*_\varepsilon(-x)\,\gamma^+_\varepsilon$$
$$c = \sum_{\varepsilon>0} \alpha_\varepsilon\,\gamma_\varepsilon + \beta^*_\varepsilon\,\gamma^+_\varepsilon$$

which diagonalizes the Hamiltonian provided that

$$\begin{cases} \varepsilon u = -iv_F u' + t\,\delta(x)\,(\alpha+\beta) \\ \varepsilon v = iv_F v' - t\,\delta(x)\,(\alpha+\beta) \\ \varepsilon\alpha = t\,[u(0)-v(0)] \\ \varepsilon\beta = t\,[u(0)-v(0)] \end{cases}$$

~~Fractional Josephson~~

The wave function corresponding to an incident electron with energy $\varepsilon$, reflected as an electron with probability amplitude $r$ and Andreev reflected as a hole with probability amplitude $r_A$ reads

$$u_\varepsilon(x) = \begin{cases} e^{i\varepsilon x/v_F} & x<0 \\ r e^{i\varepsilon x/v_F} & x>0 \end{cases}$$

$$v_\varepsilon(x) = \begin{cases} r_A e^{-i\varepsilon x/v_F} & x<0 \\ 0 & x>0 \end{cases}$$

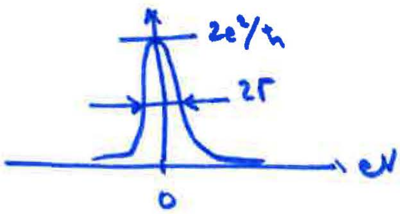
The above equations yields

$$r_A = \frac{i\Gamma}{\varepsilon + i\Gamma} \qquad \Gamma = \frac{2t^2}{v_F}$$

The Andreev current is then:

$$I = \frac{e^2}{h}\int d\varepsilon \, |r_{A\varepsilon}|^2 \left[f(\varepsilon+eV) - f(\varepsilon-eV)\right]$$

$$\frac{\partial I}{\partial V} = \frac{2e^2}{h}\,\frac{\Gamma^2}{(eV)^2+\Gamma^2} \qquad \text{at } T=0$$

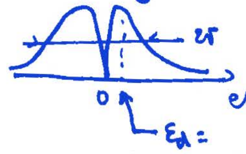



quantized ZBA

NB: + many other effects may yield (not necessarily quantized) ZBA at the edge of superconductors (trivial)

+ several aspects may explain nonquantized ZBA in topological superconductors.

NB: effect of the Majorana state on the other side of the nanowire!




$E_d$ = {energy of the ABS formed with the overlap of MBS

## Fractional Josephson effect

Let us couple two topological superconductors.
For simplicity, we consider two Kitaev models in the regime $t=\Delta$ and $\mu=0$, weakly coupled:

$$\cdots\cdots \overset{t'}{\frown} \cdots\cdots$$

The tunneling term between the chain is $H_T = t' e^{i\frac{\varphi}{2}} c_0^\dagger c_1 + h.c.$

where $c_0 \approx \frac{1}{\sqrt{2}}\Gamma_{A0}$ and $c_1 = \frac{i}{\sqrt{2}}\Gamma_{B1}$, ie $H_T \approx i t' \cos\frac{\varphi}{2}\,\Gamma_{A0}\Gamma_{B1}$

Defining $d = \frac{1}{\sqrt{2}}(\Gamma_{B1} + i\Gamma_{A0})$, $H_T = t'\cos\frac{\varphi}{2}\left(d^\dagger d - \frac{1}{2}\right)$

$\hookrightarrow$ ABS with $4\pi$ periodic energy
(energy $\to 0$ if $t' \to 0$)

$\hookrightarrow$ $4\pi$ periodic Josephson relation if the occupation of the ABS does not change

$$I(t) = P(t)\, I_0 \sin \frac{\varphi(t)}{2}$$

$\varphi(t) = 2eV_{dc}t \longrightarrow I(t) \sim \sin(eVt)$

$\varphi(t) = 2eV_{dc} + \frac{2eV_{ac}}{\omega}\sin\omega t \longrightarrow$ Shapiro steps if $eV_{dc} = n\omega$

only even steps compared with conventional JJ (cf p.13)

NB: these effects are lost if the parity is not ~~...~~ conserved

$\longleftrightarrow$ inelastic process due to $\begin{cases} \text{coupling to a bath of fermions \& bosons} \\ \text{nonadiabatic effects } (V(t) \neq 0) \end{cases}$

Interest for computation?

* $N$ fermions $\leftrightarrow$ $2N$ majorana fermions
* Parity constraint: $2^{N-1}$ ground state degeneracy
* Braiding operation:

$$\begin{cases} \gamma'_1 \propto \gamma_2 \\ \gamma'_2 \propto \gamma_1 \end{cases}$$

$\gamma_1'^2 = \gamma_2^2 = 1 \implies$ phase $\pm 1$

if $\gamma'_1 = \gamma_2$ then as the parity constraint imposes $\gamma_1\gamma_2 = \gamma'_1\gamma'_2$

we find $\gamma'_2 = \gamma'_1\gamma_1\gamma_2 = \gamma_2\gamma_1\gamma_2 = -\gamma_1$

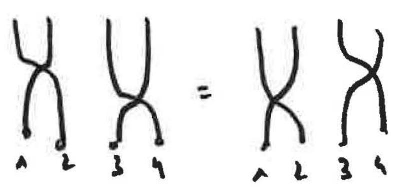
i.e $\gamma'_1 = U_{12}^{\dagger}\gamma_1 U_{12} = \frac{1}{2}(1-\gamma_1\gamma_2)\gamma_1(1+\gamma_1\gamma_2) = \frac{1}{2}(\dots)\,\gamma_2$

$\gamma'_2 = U_{12}^{\dagger}\gamma_2 U_{12} = \frac{1}{2}(1-\gamma_1\gamma_2)\gamma_2(1+\gamma_1\gamma_2) = -\gamma_1$

$U_{12} = e^{i\varphi} e^{\frac{\pi}{4}\gamma_1\gamma_2}$

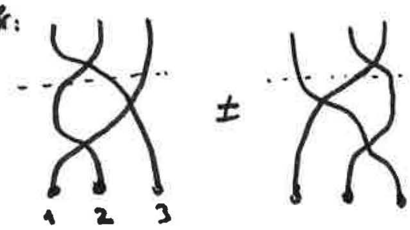
(We'll discuss later how to implement)

* non-abelian braiding:



$U_{12}U_{34} = U_{34}U_{12}$

But:



$U_{12}U_{23}U_{12} = U_{23}U_{12}U_{23}$

$(U_{12}U_{23} \neq U_{23}U_{12})$

(However not universal for quantum computation...)