Signatures of odd-frequency correlations in the Josephson current of superconductor/ferromagnet hybrid junctions

Caroline Richard,[1] Alexandre Buzdin,[2] Manuel Houzet,[1] and Julia S. Meyer[1]

[1] *Univ. Grenoble Alpes, INAC-SPSMS, F-38000 Grenoble, France and CEA, INAC-SPSMS, F-38000 Grenoble, France*

[2] *University Bordeaux, LOMA UMR-CNRS 5798, F-33405 Talence Cedex, France*

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Contacting a bilayer ferromagnet with a singlet even-frequency superconductor allows for the realization of an effective triplet odd-frequency superconductor. In this work, we investigate the Josephson effect between superconductors with different symmetries (e.g., odd versus even frequency). In particular, we study the supercurrent flowing between two triplet odd-frequency superconducting leads through a weak singlet even-frequency superconductor. We show that the peculiar temperature dependence of the critical current below the superconducting transition of the weak superconductor is a signature of the competition between odd/odd-frequency and odd/even-frequency Josephson couplings.

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I. INTRODUCTION

It is well known that superconductivity arises from the formation of Cooper pairs of electrons, where the wave function of a pair is a function of spin, space, and frequency (or time). The Pauli principle tells us that such a wave function should be antisymmetric. In dirty metals, due to multiple scattering events on impurities, the orbital part is necessarily symmetric. As a consequence, a spin-singlet pairing is even in frequency and a spin-triplet pairing is odd in frequency [1]. In conventional superconductors (S) the pairing is even in frequency. However, it has been predicted that, thanks to the proximity effect, one may induce spin-triplet odd-frequency correlations in hybrid superconducting/ferromagnetic structures (S/F). When F is homogeneous, the triplet proximity effect involves electrons of opposite spins and is short-ranged [2]. By contrast, an inhomogeneous magnetization, as, e.g., in noncollinear bilayer ferromagnets (F'/F), also induces long-range triplet correlations between electrons with parallel spins [3,4]. Furthermore, if F' is short whereas F is much longer than the coherence length of singlet correlations, "pure" triplet odd-frequency correlations are induced at the extremity of the long ferromagnet. Thus the $S/F'/F$ structure realizes an effective triplet odd-frequency reservoir (S_T). In this paper, we study how to probe these odd-frequency correlations.

Recently, long-range supercurrents have been measured in trilayer ferromagnetic Josephson junctions, which can be viewed as Josephson junctions between two odd-frequency reservoirs (S_T/S_T junctions) [5–7]. While this indicates the presence of triplet odd-frequency correlations, the measurements did not present any peculiarities as compared to observations made in "classic" Josephson junctions connecting two conventional superconductors (S/S). Indeed, from a symmetry point of view the S_T/S_T junction as well as the S/S junction realize a coupling between two reservoirs sharing the same symmetry (odd/odd frequency for S_T/S_T and even/even frequency for S/S), yielding similar supercurrent measurements. By contrast, the current-phase relation of S_T/S junctions is predicted to be superharmonic [8–10]. This specificity originates from the odd/even-frequency Josephson coupling. Namely, the symmetry mismatch between the reservoirs prohibits mechanisms involving the transfer of a single Cooper pair. Instead, the supercurrent originates from the coherent flow of an even number of pairs, yielding a peculiar π-periodic current-phase relation.

In this work, we explore the competition between odd/odd-frequency and odd/even-frequency Josephson couplings through the temperature dependence of the critical current of hybrid junctions. In particular, we study the current through an $S_T/S/S_T$ junction, where a conventional superconductor of bare critical temperature T_c is sandwiched between two effective triplet odd-frequency reservoirs. Such a junction may be realized in a $S'/F'/F/S/F/F'/S'$ hybrid junction, where S' are conventional superconductors with a critical temperature T_c'; see Fig. 1.

In the following, we assume that $T_c' \gg T_c$. Above the critical temperature T_c of the superconducting layer, an effective odd/odd-frequency Josephson coupling builds up: the transfer of "odd-frequency" pairs between leads happens via virtual Andreev pairs in the island. Below T_c, when the central S layer is superconducting, the quasiparticles above the gap coexist with the even-frequency condensate of Cooper pairs. Therefore, an additional odd/even-frequency Josephson coupling is generated at the interfaces between the layer and the leads, generating a double S_T/S Josephson junction.

We will show that the currents associated with odd/odd-frequency and odd/even-frequency Josephson couplings are in competition. Besides a peculiar current-phase relation, this leads to a suppression of the critical current below the transition temperature of the weak superconductor.

The outline of the paper is as follows. In Sec. II, we introduce the formalism and, in Sec. III, we derive the Green function of an effective S_T reservoir. Then, in Sec. IV, we treat the full $S_T/S/S_T$ junction and compute both its current-phase relation and its critical current. Finally, we briefly discuss metallic junctions in Sec. V, before concluding in Sec. VI.

II. FORMALISM

Within the quasiclassical theory, the equilibrium properties of hybrid superconducting/ferromagnetic junctions can be expressed via the quasiclassical Matsubara Green function g, which is a 4×4 matrix in the particle-hole and spin spaces,

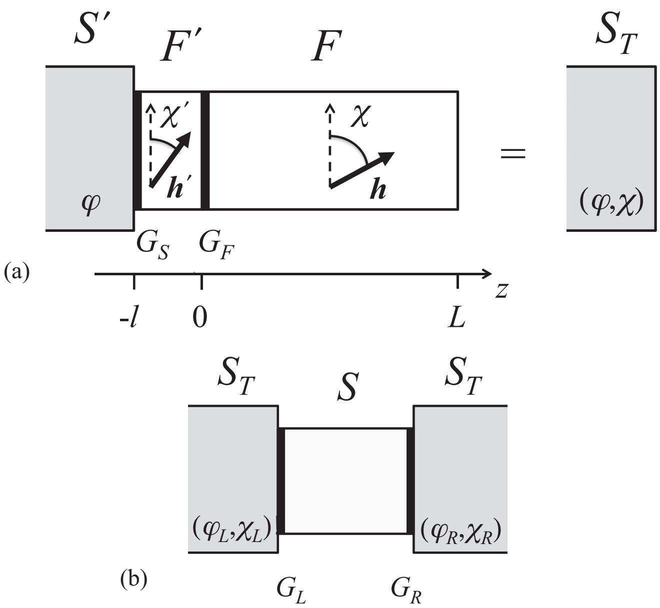

FIG. 1. (a) Realization of an effective triplet odd-frequency superconductor S_T by contacting a ferromagnetic F'/F bilayer with noncollinear magnetizations, with a relative angle $\theta = \chi - \chi'$ between their directions, to a singlet even-frequency superconductor S'. (b) Effective $S_T/S/S_T$ double Josephson junction formed with triplet odd-frequency superconducting leads connected through a singlet even-frequency superconducting layer.

and obeys the normalization conditions $g^2 = 1$ and $\mathrm{Tr}[g] = 0$. Within circuit theory [11], g takes the constant value g_i in each superconducting or ferromagnetic node or reservoir of the circuit. Thus the circuit theory assumes that the thickness of each layer, modeled as a node, is much shorter than the coherence length for superconducting correlations in that layer. The Green functions in the nodes obey the equations

$$\frac{2\pi G_Q}{\delta_i}[(\omega + i\boldsymbol{h}_i \cdot \boldsymbol{\sigma})\tau_z + \hat{\Delta}_i, g_i] + \sum_j \hat{I}_{ij} = 0. \quad (1)$$

Here, $\delta_i = 1/(\nu_0 V_i)$ is the mean level spacing in node i, with volume V_i and density of states ν_0 in the normal metallic phase, $G_Q = e^2/\pi$ is the conductance quantum, $\omega = (2n+1)\pi T$ is a positive Matsubara frequency at temperature T ($n \geqslant 0$), $\boldsymbol{h}_i$ is an exchange field acting on the electron spin in ferromagnetic nodes, and $\hat{\Delta}_i = \Delta_i(\cos\varphi_i \tau_x - \sin\varphi_i \tau_y)$, where Δ_i and φ_i are the modulus and phase of the superconducting order parameter in superconducting nodes. The Green functions at negative Matsubara frequencies can be obtained from the analyticity condition,

$$g_i(-\omega) = -\tau_z g_i^\dagger(\omega)\tau_z, \quad (2)$$

or the particle-hole symmetry,

$$g_i(-\omega) = \sigma_y \tau_x g_i^T(\omega)\sigma_y \tau_x. \quad (3)$$

The Pauli matrices τ_i and σ_i ($i = x,y,z$) act in particle-hole and spin space, respectively. Moreover, the spectral current between two nodes or leads i and j,

$$\hat{I}_{ij} = \frac{G_{ij}}{2}[g_j, g_i], \quad (4)$$

is related with the normal-state conductance G_{ij} of the “connector” between them. The current flowing through that connector is

$$I_{ij} = -\frac{\pi T}{2e}\mathrm{Im}\sum_{\omega>0}\mathrm{Tr}[\tau_z \hat{I}_{ij}]. \quad (5)$$

Finally, in the case of a superconducting node with “bare” critical temperature T_{ci}, the order parameter should satisfy the self-consistency equation

$$\ln\frac{T_{ci}}{T} = 2\pi T\sum_{\omega>0}\left(\frac{1}{\omega} - \frac{e^{-i\varphi_i}}{4\Delta_i}\mathrm{Tr}[\tau_-(g_i + g_i^\dagger)]\right), \quad (6)$$

where $\tau_\pm = (\tau_x \pm i\tau_y)/2$.

Note that Eq. (6) is complex: it combines two real equations, one for the amplitude, $|\Delta_i|$, of the order parameter in node i and another one for its phase, φ_i. The latter is equivalent to the current conservation condition in that node. Namely, combining Eqs. (1) and (6), we find that

$$\begin{aligned}\sum_j I_{ij} &= \frac{\pi^2 T G_Q}{e\delta_i}\mathrm{Im}\sum_{\omega>0}\mathrm{Tr}[\tau_z[\hat{\Delta}_i, g_i]] \\ &= -\frac{2\pi^2 T G_Q \Delta_i}{e\delta_i}\mathrm{Im}\sum_{\omega>0}e^{-i\varphi_i}\mathrm{Tr}[\tau_-(g_i + g_i^\dagger)] = 0\end{aligned} \quad (7)$$

is automatically satisfied.

III. TRIPLET ODD-FREQUENCY RESERVOIRS

As mentioned in the Introduction, a trilayer structure consisting of a conventional superconductor and two noncollinear ferromagnets realizes an effective triplet reservoir ($S_T \equiv S'/F'/F$); see Fig. 1(a). As it was demonstrated in Ref. [13], the circuit theory of Sec. II can be extended to describe the long-range triplet proximity effect in a ferromagnetic node whose thickness is much larger than the ferromagnetic coherence length associated with short-range superconducting correlations, but shorter than the coherence length associated with long-range ones. In this section, we use a similar approach to derive the Green function of S_T. We take the z axis perpendicular to the layers and choose the magnetizations of both ferromagnetic layers to lie in the xy plane, namely

$$\begin{aligned}\boldsymbol{h}' &= h'(\cos\chi'\hat{x} + \sin\chi'\hat{y}) \text{ in } F', \\ \boldsymbol{h} &= h(\cos\chi\hat{x} + \sin\chi\hat{y}) \text{ in } F.\end{aligned} \quad (8)$$

Hence $\xi_F' = \sqrt{D/h'}$ and $\xi_F = \sqrt{D/h}$ are the ferromagnetic coherence lengths in F' and F, respectively.

The first layer F' generates only short-range correlations. Thus its length l should not exceed the ferromagnetic coherence length ξ_F'. By contrast, the noncollinear second layer F generates triplet correlations with all different spin projections. To filter out only the long-range components, its length L needs to be much longer than ξ_F.

Within the quasiclassical theory, we call g_S, $g_{F'}$, and g_F the Green function in the S', F', and F layers, respectively. Here, S' is a reservoir. Thus, in the subgap regime, $\omega \ll T_c'$, the Green function g_S takes the form $g_S = \cos\varphi\tau_x - \sin\varphi\tau_y \equiv \tau_\varphi$, where φ is the superconducting phase. The Green functions in the ferromagnetic layers will be determined in the following, using the formalism introduced in Sec. II. In particular, the Green function g_T of our effective triplet reservoir is related to correlations developing at the edge of F, namely $g_T = g_F(L)$.

Assuming $l \ll \xi'_F$, we can use circuit theory, where the F' layer is a ferromagnetic node. Its Green function $g_{F'}$ obeys

$$\left[\frac{2\pi G_Q}{\delta_{F'}}(\omega + ih'\sigma_{\chi'})\tau_z + \frac{1}{2}(G_S g_S + G_F g_F), g_{F'}\right] = 0, \quad (9)$$

where $\delta_{F'}$ is the mean level spacing in F' and G_S (G_F) is the conductance of the S'/F' (F'/F) interface. Furthermore, we introduced the short-hand notation $\sigma_{\chi'} = \cos\chi'\sigma_x + \sin\chi'\sigma_y$. In the following, we assume that F' is more strongly coupled to S', i.e., $G_S \gg G_F$, and neglect the leakage current at the F'/F interface to obtain

$$g_{F'} = \frac{(\omega + ih'\sigma_{\chi'})\tau_z + \gamma_S\tau_\varphi}{\sqrt{(\omega + ih'\sigma_{\chi'})^2 + \gamma_S^2}}. \quad (10)$$

Here $\gamma_S = \delta_{F'} G_S/(2\pi G_Q)$ is the induced minigap in F'. Note that the same Green function with $\gamma_S = \Delta > h' = E_Z$ describes a superconductor subject to an external Zeeman field E_Z. The advantage of using an S'/F' bilayer is the possibility of realizing both regimes $h' < \gamma_S$ and $h' > \gamma_S$ by tuning, e.g., the transparency of the S'/F' interface (G_S) or the thickness of the F' layer ($\delta_{F'} \propto 1/l$).

Having determined $g_{F'}$, we now turn to the long ferromagnetic layer F of length $L \gg \xi_F$. Close to the F'/F interface, both short- and long-range correlations coexist. The fast oscillatory behavior of the short-range correlations prevents us from directly applying the circuit theory. However, within a few ξ_F from the F'/F interface, the short-range correlations are suppressed, and only the nonoscillating long-range triplet correlations survive. Then, for $\xi_F \ll z \ll \xi_N$ with $\xi_N = \sqrt{D/2\pi T}$, we find $g_F(z) \approx g_T = \text{const}$. In particular, if $\xi_F \to 0$, the Green function may be considered constant throughout the layer. Thus, within a circuit theory approach, the long F layer maps to a ferromagnetic node with $\xi_F \to 0$ or, correspondingly, $h \to \infty$. Consequently g_T obeys the equation,

$$[\omega\tau_z + ih\sigma_\chi\tau_z + \gamma_F g_{F'}, g_T] = 0 \quad \text{with } h \to \infty, \quad (11)$$

where $\gamma_F = \delta_F G_F/(2\pi G_Q)$, and δ_F is the mean level spacing in F.

To determine g_T, we orthogonally decompose the Green function $g_{F'}$ with respect to the length scale over which it decays in F'. Namely, we write $g_{F'} = g_\parallel + g_\perp$, where $[g_\parallel, \sigma_\chi\tau_z] = 0$ and $\{g_\perp, \sigma_\chi\tau_z\} = 0$. Here $g_\parallel$ contains the long-range correlations, whereas $g_\perp$ contains the short-range correlations in F. Then, $g_\parallel$ may be further decomposed by noting that terms $\propto\sigma_\chi\tau_z$ can be absorbed into h in Eq. (11). Thus we write $g_\parallel$ as $g_\parallel = \tilde{g}_\parallel + J\sigma_\chi\tau_z$ with $J = \frac{1}{4}\text{Tr}[\sigma_\chi\tau_z g_\parallel]$.

In the limit $h \to \infty$, the short-range correlations are completely suppressed, while the long-range correlations are not affected by h. As shown in the AppendixA, Eq. (11) may be rewritten in the form

$$[\omega\tau_z + \gamma_F\tilde{g}_\parallel, g_T] = 0, \quad (12)$$

where $\tilde{g}_\parallel(\omega) = \alpha(\omega)\tau_z + i\beta(\omega)\sigma_\chi\tau_\varphi$ can be obtained from Eq. (10) with

$$\alpha(\omega) = \frac{1}{2}\sum_\pm \frac{(\omega \pm ih')}{\sqrt{(\omega \pm ih')^2 + \gamma_S^2}}, \quad (13)$$

$$\beta(\omega) = -\frac{i}{2}\sin\theta\sum_\pm \frac{\pm\gamma_S}{\sqrt{(\omega \pm ih')^2 + \gamma_S^2}}, \quad (14)$$

where $\theta = \chi - \chi'$ is the relative angle between the magnetization directions of F and F'.

Finally, the Green function for the effective triplet reservoir solving Eq. (12) reads

$$g_T = \cosh\vartheta\,\tau_z + i\,\sinh\vartheta\,\sigma_\chi\tau_\varphi, \quad (15)$$

with

$$\cosh\vartheta(\omega) = \frac{\omega + \gamma_F\alpha(\omega)}{\sqrt{[\omega + \gamma_F\alpha(\omega)]^2 - \gamma_F^2\beta^2(\omega)}}, \quad (16)$$

$$\sinh\vartheta(\omega) = \frac{\gamma_F\beta(\omega)}{\sqrt{[\omega + \gamma_F\alpha(\omega)]^2 - \gamma_F^2\beta^2(\omega)}}. \quad (17)$$

The Green function of the effective triplet reservoir is thus described by a single angle ϑ which depends, however, on all the parameters $(h', \gamma_S, \gamma_F, \theta)$. Note that $\cosh\vartheta$ corresponds to the normal Green function and encodes the density of states, whereas $\sinh\vartheta$ corresponds to the anomalous Green function, describing the induced triplet correlations. As $\beta \propto \sin\theta$, we see that the triplet correlations vanish for collinear F'/F layers ($\theta = 0\,[\pi]$) as expected, while they are maximal for perpendicular magnetizations ($\theta = \pi/2$). For simplicity, we will consider only the case $\theta = \pi/2$ in the following. The generalization to arbitrary angles is straightforward.

Knowing the Green function of the effective triplet reservoir, we can now obtain its density of states (DOS),

$$\nu(\epsilon) = \nu_0\text{Re}[\cosh\vartheta(-i\epsilon + 0^+)], \quad (18)$$

where ν_0 is the density of states of the normal metal. As the DOS is even in ϵ, we will consider positive energies, $\epsilon > 0$, only.

The functions $\alpha(-i\epsilon)$ and $\beta(-i\epsilon)$ possess singularities at $\epsilon = E_c^\pm \equiv |h' \pm \gamma_S|$, which are inherited by the DOS. We will concentrate on the limiting cases $h' \ll \gamma_S$ and $h' \gg \gamma_S$, when these singularities are far away from $\epsilon = 0$. In particular, for $\epsilon, \gamma_F, h' \ll \gamma_S$, we find

$$\nu(\epsilon) \approx \nu_0\left[1 + \frac{1}{2}\left(\frac{\gamma_F h'}{\gamma_S^2}\right)^2\left(1 + 3\frac{\epsilon^2}{\gamma_S^2}\right)\right]. \quad (19)$$

Thus the zero-energy DOS is enhanced as compared to the normal state. Furthermore, it displays a broad dip at $\epsilon = 0$. In the opposite regime, for $\epsilon, \gamma_F, \gamma_S \ll h'$, we find

$$\nu(\epsilon) \approx \nu_0\left[1 + \frac{1}{2}\left(\frac{\gamma_S}{h'}\right)^2\frac{1 - \frac{\epsilon^2}{\gamma_F^2}}{\left(1 + \frac{\epsilon^2}{\gamma_F^2}\right)^2}\right]. \quad (20)$$

Here as well, the zero-energy DOS is enhanced. However, it possesses a narrow peak at $\epsilon = 0$. The enhancement of the DOS with respect to its value in the normal state, as well as a peak at $\epsilon = 0$, were discussed in similar models of $S/F'/F$ structures with large exchange fields [12–16].

Similarly, we may analyze the triplet correlations encoded in $\sinh\vartheta(\omega)$. The symmetry relation (3) together with Eq. (15) readily yield that the triplet correlations are odd in frequency, $\sinh\vartheta(-\omega) = -\sinh\vartheta(\omega)$. Furthermore, for $\gamma_F, h' \ll \gamma_S$,

$$\sinh\vartheta(\omega) \approx -\frac{\gamma_F h'}{\gamma_S^2}\frac{1}{\left(1+\frac{\omega^2}{\gamma_S^2}\right)^{3/2}}. \quad (21)$$

Thus the correlations decay on the energy scale γ_S. By contrast, for $\gamma_F \ll \gamma_S \ll h'$,

$$\sinh\vartheta(\omega) \approx -\frac{\gamma_S}{h'}\frac{1}{1+\frac{\omega}{\gamma_F}}\frac{1}{1+\frac{\omega^2}{h'^2}}. \quad (22)$$

In that case, the correlations are reduced as soon as $\omega > \gamma_F$ and then decay more rapidly on the energy scale h'.

IV. $S_T/S/S_T$ JUNCTION

A. Current-phase relation

We are now in a position to study the effective $S_T/S/S_T$ junction presented in the Introduction; see Fig. 1(b). Within circuit theory, S is a superconducting node of bare critical temperature T_c and mean level spacing δ. It is connected to a left and a right effective triplet reservoir (S_T) via connectors of conductances G_L and G_R, respectively. Then g, g_L, and g_R are the Green function in the node, the left reservoir, and the right reservoir, respectively. Assuming $G_L, G_R \ll G_F$, we can neglect the inverse proximity effect and use rigid boundary conditions in the reservoirs. Thus g_L and g_R are the Green functions derived in the previous section, whereas g will be determined in the following.

For simplicity, here we consider the effective odd-frequency triplet reservoirs to be identical by choosing $\vartheta_L(\omega) = \vartheta_R(\omega) = \vartheta(\omega)$ and $G = G_L = G_R$. However, the reservoirs may have different superconducting phases $\varphi_{L/R}$ and magnetization axes $\chi_{L/R}$. Assuming that all magnetizations lie in the same plane [17], we choose $\varphi_{L/R} = \pm\varphi/2$ and $\chi_{L/R} = \pm\chi/2$, such that φ is the phase bias of the junction, whereas χ is the relative angle between the magnetization axes. Then $g_{L/R}$ may be written as

$$g_{L/R} = \cosh\vartheta\,\tau_z + i\,\sinh\vartheta\,\sigma_{\pm\chi/2}\tau_{\pm\varphi/2}. \quad (23)$$

According to Eq. (1), g obeys

$$\left[\omega\tau_z + \Delta\tau_\phi + \gamma\frac{g_L+g_R}{2}, g\right] = 0, \quad (24)$$

where $\gamma = \delta G/(2\pi G_Q)$. Furthermore, Δ and ϕ are the amplitude and the phase of the order parameter in S, satisfying Eq. (6).

We concentrate on the weak coupling regime, $\gamma \ll T$, where it is possible to perform a perturbative expansion of g around its bulk value g_0. To this end, we write $g = g_0 + g_1 + \cdots$, where $g_1 \ll g_0$. Accordingly, the charge current may be written in the form $I_{L/R} = I^{(1)}_{L/R} + I^{(2)}_{L/R} + \cdots$, where

$$I^{(i)}_{L/R} = \frac{G}{2e}\pi T\,\mathrm{Im}\sum_{\omega>0}\frac{1}{2}\mathrm{Tr}[\tau_z[g_{i-1}, g_{L/R}]]. \quad (25)$$

The bare Green function of the superconducting node S reads

$$g_0 = \frac{\omega\tau_z + \Delta_0\tau_\phi}{\sqrt{\omega^2+\Delta_0^2}}, \quad (26)$$

where $\Delta_0(T)$ solves the standard BCS equation,

$$\ln\frac{T_c}{T} = 2\pi T\sum_{\omega>0}\left(\frac{1}{\omega} - \frac{1}{\sqrt{\omega^2+\Delta_0^2}}\right), \quad (27)$$

while the phase ϕ is undetermined for the bare node. However, the $U(1)$ symmetry is broken once the node is coupled to the reservoirs.

Incorporating g_0 in Eq. (25), we obtain $I^{(1)} = 0$. Namely, no Josephson coupling exists at first order. Indeed, due to the symmetry mismatch between the effective triplet reservoirs and the singlet superconducting dot, a single Cooper pair may not carry a current. We thus turn to the next order and compute g_1. It obeys the first order expansion of Eq. (24) supplemented by the normalization condition, namely,

$$[\omega\tau_z + \Delta_0\tau_\phi, g_1] = -\left[\gamma\frac{g_L+g_R}{2} + \Delta_1\tau_\phi, g_0\right], \quad (28)$$

$$\{g_0, g_1\} = 0. \quad (29)$$

Here Δ_1 is the first order correction to Δ_0. The system is solved by

$$g_1 = -\frac{1}{2\sqrt{\omega^2+\Delta_0^2}}\left[\frac{\gamma}{2}(g_L+g_R) + \Delta_1\tau_\phi, g_0\right]g_0.$$

Additionally, the self-consistency equation (6) yields

$$\frac{\Delta_1}{\Delta_0} = -\gamma\left(\sum_\omega\frac{\omega\,\cosh\vartheta(\omega)}{\left(\omega^2+\Delta_0^2\right)^{3/2}}\right)\Bigg/\left(\sum_\omega\frac{\Delta_0^2}{\left(\omega^2+\Delta_0^2\right)^{3/2}}\right). \quad (30)$$

Note that $\Delta_1/\Delta_0 < 0$, i.e., superconductivity is weakened by the coupling. This reduction does not depend on the phase bias φ, and may be attributed to the gapless property of the odd-frequency triplet reservoirs that was discussed in the end of Sec. III; cf. Eqs. (19) and (20). It features an inverse proximity effect, where the quasiparticles, existing at zero energy in the leads, weaken superconductivity in the node S. By consequence, the effective critical temperature T_c^* of S at finite γ is decreased,

$$\frac{T_c^* - T_c}{T_c} \approx -2\pi T\sum_{\omega>0}\frac{\gamma\,\cosh\vartheta(\omega)}{\omega^2} < 0. \quad (31)$$

Note that a dependence of T_c^* on the relative orientation of the magnetizations in adjacent layers would arise in higher order in γ.

Then, incorporating g_1 in Eq. (25), we find

$$I^{(2)} = -\frac{\gamma G}{2e}\{a(T)\cos\chi\,\sin\varphi - b(T)[\sin\varphi\,\cos(2\phi) - (\cos\chi + \cos\varphi)\sin(2\phi)]\}, \quad (32)$$

where

$$a(T) = \pi T \sum_{\omega>0} \frac{\sinh^2 \vartheta(\omega)}{\sqrt{\omega^2+\Delta_0^2}}\left(2 - \frac{\Delta_0^2}{\omega^2+\Delta_0^2}\right), \tag{33}$$

$$b(T) = \pi T \sum_{\omega>0} \frac{\sinh^2 \vartheta(\omega)\,\Delta_0^2}{\left(\omega^2+\Delta_0^2\right)^{3/2}}. \tag{34}$$

The first line of (32) may be identified as a quasiparticle current between the two triplet reservoirs that does not depend on the phase of the central island. By contrast, the second line of (32) may be identified as a condensate contribution, corresponding to a superharmonic Josephson effect between the triplet reservoirs and the singlet central island, that depends on the phase ϕ. A comprehensive discussion of the quasiparticle and different contributions to the condensate currents is given in Sec. V on metallic junctions.

Current conservation or, equivalently, the self-consistency equation (6) fixes the phase $\phi = k\pi/2$ with $k \in \mathbb{Z}$. Furthermore, energy minimization selects which of those phases corresponds to stable solutions and imposes

$$\phi = \begin{cases} 0 & \text{if } \cos\varphi + \cos\chi > 0, \\ \pi/2 & \text{otherwise.} \end{cases} \tag{35}$$

As a consequence of the odd/even-frequency Josephson coupling between two triplet/singlet pairs, ϕ is defined modulo π instead of 2π.

Inserting Eq. (35) into (32), we obtain [20]

$$I^{(2)} = -\frac{\gamma G}{2e}[a(T)\cos\chi - b(T)\mathrm{sgn}(\cos\chi + \cos\varphi)]\sin\varphi. \tag{36}$$

While the quasiparticle contribution is a continuous function of the phase bias φ, the condensate contribution displays a jump at $\cos\chi + \cos\varphi = 0$ [21]. Both $a(T), b(T) \geqslant 0$. Thus, when $\chi < \pi/2$ ($\chi > \pi/2$), the two contributions are opposed for phases $\varphi < \pi - \chi$ ($\varphi < \pi - \chi$), whereas they have the same direction for phases $\varphi > \pi - \chi$ ($\varphi < \pi - \chi$). Examples of typical current-phase relations are shown in Fig. 2.

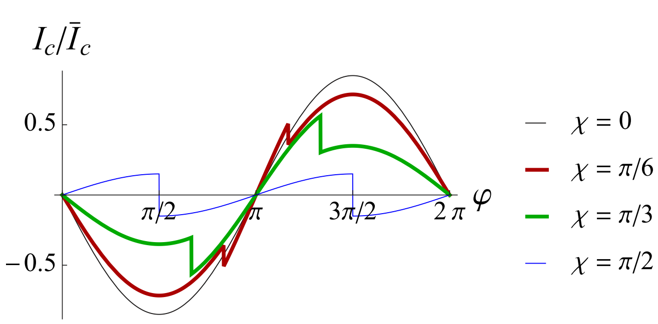

FIG. 2. (Color online) Typical current-phase relations. The current is plotted in units of $\bar{I}_c = \gamma G a(T)/(2e)$ for different magnetization angles $\chi = 0, \pi/6, \pi/3$, and $\pi/2$ (bottom to top at $\varphi < \pi/2$) and $b(T)/a(T) = 0.15$; cf. Eq. (36). The critical current is achieved at phase $\varphi = \pi/2$ for $\chi = 0$ and $\pi/6$, and at phase $\varphi = \pi - \chi$ for $\chi = \pi/3$ and $\pi/2$.

B. Critical current

To get more insight into the competition between the condensate and quasiparticle contributions, we study the critical current I_c of the junction, where $I_c(T,\chi) = \max_\varphi[I^{(2)}(T,\varphi,\chi)]$. Based on the considerations above, the critical current is achieved for phase bias $\varphi = \pi/2$ or $\varphi = \pi - \chi$. Namely, $I_c(T,\chi) = \max[I_1, I_2]$, where

$$I_1(T,\chi) = |I^{(2)}(T,\pi/2,\chi)| = \frac{\gamma G}{2e}|a(T)|\cos\chi| - b(T)|,$$

$$I_2(T,\chi) = |I^{(2)}(T,\pi-\chi,\chi)| = \frac{\gamma G}{2e}[a(T)|\cos\chi| + b(T)]\sin\chi.$$

For a fixed χ, as a function of temperature, the critical current I_c lies either on the I_1 or on the I_2 branch. While the I_2 branch increases monotonously with decreasing temperature, the temperature dependence of the I_1 branch is more complicated. Above T_c^*, the I_1 branch increases as $\ln(\gamma_S/T)$, for $h', T \ll \gamma_S$, and as $(\gamma_F/T)^2$, for $\gamma_F \ll \gamma_S, T \ll h'$, with decreasing temperature. At T_c^*, it has a cusp. When further decreasing temperature below T_c^*, it increases much more slowly or even decreases. For $\gamma_S \gg h'$ and $T \lesssim T_c^*$, we find

$$I_1(T,\chi) \simeq \frac{\gamma G}{2e}\left(\frac{\gamma_F h'}{\gamma_S^2}\right)^2 \times \left\{|\cos\chi|\ln\frac{\gamma_S}{T} - \frac{(T_c^*-T)}{T_c^*}(2|\cos\chi|+1)\right\}, \tag{37}$$

which decreases with decreasing temperature for all values of χ. By contrast, for $\gamma_S \ll h'$ and $T \lesssim T_c^*$, we find

$$I_1(T,\chi) \simeq \frac{7\gamma G}{8\pi^2 e}\zeta(3)\left(\frac{\gamma_S}{h'}\right)^2\left(\frac{\gamma_F}{T}\right)^2 \times \left\{|\cos\chi| - \mathcal{N}\frac{(T_c^*-T)}{T_c^*}(2|\cos\chi|+1)\right\}, \tag{38}$$

where $\mathcal{N} = 31\zeta(5)/[7\zeta(3)]^2 \approx 0.5$, which slowly increases with decreasing temperature for angles $\chi \lesssim \pi/3$.

Which branch the critical current follows is determined by the ratio $b(T)/a(T)$. We find that the ratio $b(T)/a(T)$ is zero above T_c^* and increases monotonously below T_c^*, satisfying $b(T)/a(T) < 1$, see Fig. 3.

As a consequence, at high temperatures, the critical current follows the I_1 branch. At lower temperatures, one may distinguish two different behaviors depending on whether $I_1(0,\chi)$ is larger or smaller than $I_2(0,\chi)$. The critical angle χ_c at which one switches between the two cases is given by

$$\frac{b(0)}{a(0)} = \frac{|\cos\chi_c|(1-\sin\chi_c)}{1+\sin\chi_c}. \tag{39}$$

The solution χ_c of this equation increases from 0 to $\pi/2$ as $b(0)/a(0)$ decreases from 1 to 0; cf. the dependence of the right-hand side (RHS) of Eq. (39) as a function of χ_c in Fig. 4. For angles $\chi < \chi_c$, the critical current lies on the I_1 branch at all temperatures. By contrast, for angles $\chi > \chi_c$, the current

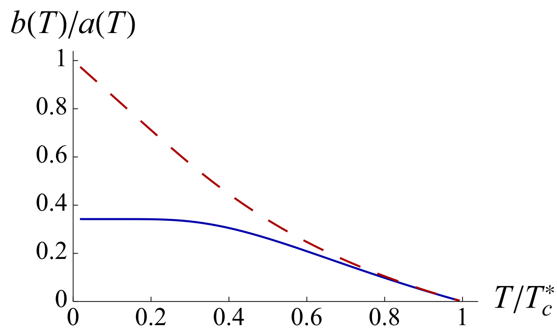

FIG. 3. (Color online) Plot of $b(T)/a(T)$ as a function of T/T_c^*, for $h',T_c^* \ll \gamma_S$ (solid line) and $\gamma_S,T_c^* \ll h'$ (dashed line).

switches to the I_2 branch at the temperature T_{12} determined by

$$\frac{b(T_{12})}{a(T_{12})} = \frac{|\cos\chi|(1-\sin\chi)}{1+\sin\chi}. \tag{40}$$

Using these considerations, we now consider the temperature dependence of the critical current for the cases $h' \ll \gamma_S$ and $h' \gg \gamma_S$, assuming $\gamma_F \ll T_c^* \ll \max[\gamma_S,h']$.

For $h' \ll \gamma_S$, $b(0)/a(0)$ decreases as $1/\ln(\gamma_S/T_c^*)$ with decreasing T_c^*/γ_S. Thus the critical angle χ_c increases. Since I_1 decreases with decreasing temperature below T_c^*, the critical current displays a unique maximum at temperatures close to T_c^* for a wide range of angles. This nonmonotonous temperature dependence provides a clear signature of the competition between odd/odd- and odd/even-frequency couplings.

By contrast, for $\gamma_S \ll h'$, we obtain $b(0)/a(0) \approx 1$. Thus a finite temperature T_{12} below which the critical current starts rising again rapidly exists for all angles. In the intermediate temperature regime $T_{12} < T < T_c^*$, the current increases very slowly. Though less pronounced than in the opposite parameter regime, this peculiar temperature dependence is a signature of the competition between the different symmetry couplings.

The temperature dependence of the critical current described above is illustrated in Figs. 5 and 6, for the cases $h' \ll \gamma_S$ and $h' \gg \gamma_S$, respectively, and for different angles χ.

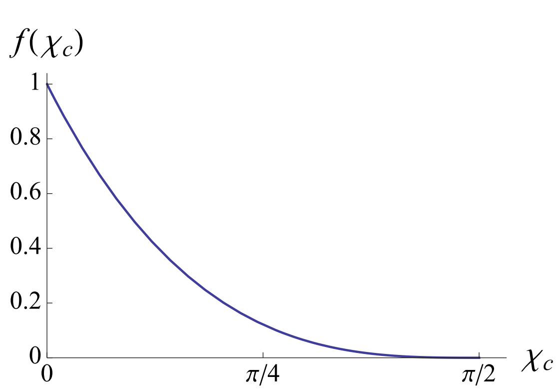

FIG. 4. (Color online) Plot of the RHS of Eq. (39), $f(\chi_c) = |\cos\chi_c|(1-\sin\chi_c)/(1+\sin\chi_c)$, as a function of χ_c.

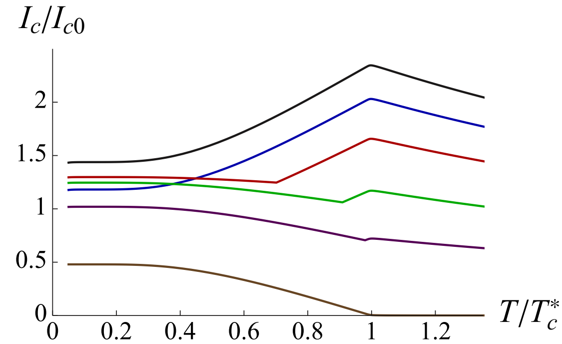

FIG. 5. (Color online) Plot of the critical current I_c [in units of $I_{c0} = \gamma G(\gamma_F h'/\gamma_S^2)^2/(2e)$] as a function of T/T_c^*, for $h',T_c^* \ll \gamma_S$, and different angles $\chi = 0,\pi/6,\pi/4,\pi/3,2\pi/5,\pi/2$ (from top to bottom at $T > T_c^*$).

V. METALLIC JUNCTION

While our calculations are limited to tunnel contacts, we may generalize our considerations to metallic junctions based on the identification of the different contributions to the current. To do so, we examine the corresponding terms in the Josephson energy. The current is then obtained by taking a derivative with respect to phase. As before, we assume that the angle between the magnetizations of adjacent ferromagnets is $\theta = \pi/2$; cf. [20].

For a metallic $S'/F'/F/S/F/F'/S'$ junction, the quasiparticle contribution near the critical temperature of the S layer takes the form

$$E_J^{\rm qp} = g\left(\Delta' - c^{\rm qp}\frac{\Delta^2(T)}{T_c^*}\right)e^{-L_S/\xi_S}\cos\chi\,\cos(\varphi_L-\varphi_R), \tag{41}$$

where $g \sim G/G_Q$, G is the normal-state conductance of the junction, and $c^{\rm qp}$ is a numerical factor of the order of unity. Here we write the phases φ_L and φ_R of the left and right superconductors S' explicitly. Furthermore, L_S and $\xi_S \sim \sqrt{D/T_c^*}$ are the length and coherence length of the

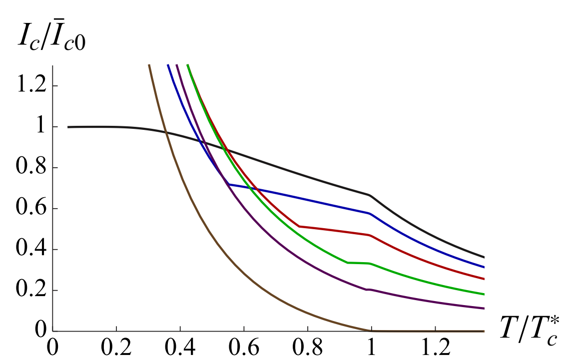

FIG. 6. (Color online) Plot of the critical current I_c [in units of $\bar{I}_{c0} = \gamma G[\gamma_F\gamma_S/(h'\Delta_0^*)]^2/(2e)$], with $\Delta_0^* \approx 1.76T_c^*$, as a function of T/T_c^*, for $\gamma_F \ll \gamma_S,T_c^* \ll h'$, and different angles $\chi = 0,\pi/6,\pi/4,\pi/3,2\pi/5,\pi/2$ (from top to bottom at $T > T_c^*$). Note that the critical current saturates at $T \lesssim \gamma_F$ (not visible on the scale of the figure).

central superconductor, respectively, where D is the diffusion coefficient. The first term $\propto\Delta'$ describes the usual Josephson energy when the central superconductor is in the normal state. It increases monotonously as the temperature decreases below the critical temperature T_c' of the leads and saturates at temperatures $T \ll T_c'$. Thus, close to $T_c^* \ll T_c'$, we may neglect its temperature dependence [22]. The second term $\propto\Delta^2(T)/T_c^*$ accounts for the reduction of the quasiparticle contribution (responsible for the *triplet* supercurrent flow) due to the developing of *singlet* superconducting correlations below T_c^*, when $\Delta(T) \propto \sqrt{T_c^* - T}\,\theta(T_c^* - T)$ is finite.

The condensate contribution consists of three different terms. Namely, there is a Josephson coupling between the left superconductor and the central superconductor, depending on the phase difference $\varphi_L - \phi$, as well as a Josephson coupling between the central superconductor and the right superconductor, depending on the phase difference $\phi - \varphi_R$. As the first harmonic in a bilayer junction is short-ranged, only the second harmonic survives for both of these contributions. Furthermore, there is a crossed term, where two pairs from each of the outer superconductors recombine in the central superconductor. Thus this contribution depends on the phase $\varphi_L + \varphi_R - 2\phi$. It is suppressed with the length of the central superconductor on the scale of the coherence length, and it depends on the angle between the magnetizations of the left and right ferromagnet. As a consequence, the condensate contribution takes the form

$$E_J^{\mathrm{cond}} = -c^{\mathrm{cond}} g \frac{\Delta^2(T)}{T_c^*}\{\cos[2(\varphi_L - \phi)] + \cos[2(\phi - \varphi_R)] + 2e^{-L_S/\xi_S}\cos\chi\,\cos(\varphi_L + \varphi_R - 2\phi)\}, \tag{42}$$

where $c^{\mathrm{cond}} \propto \xi_F/L$ [10]. With $\varphi_L = -\varphi_R = \varphi/2$, the expression simplifies to

$$E_J^{\mathrm{cond}} = -2c^{\mathrm{cond}} g \frac{\Delta^2(T)}{T_c^*}\left\{\cos\varphi + e^{-\frac{L_S}{\xi_S}}\cos\chi\right\}\cos(2\phi). \tag{43}$$

Then, minimization of the energy with respect to ϕ yields

$$\phi = \begin{cases} 0 & \text{if } \cos\varphi + e^{-L_S/\xi_S}\cos\chi > 0, \\ \pi/2 & \text{otherwise.} \end{cases} \tag{44}$$

Finally, the supercurrent accounting for both the quasiparticle and condensate contributions reads

$$I = -G\left\{\left[\Delta' - c^{\mathrm{qp}}\frac{\Delta^2(T)}{T_c^*}\right]e^{-L_S/\xi_S}\cos\chi - 2c^{\mathrm{cond}}\frac{\Delta^2(T)}{T_c^*}\mathrm{sgn}(\cos\varphi + e^{-L_S/\xi_S}\cos\chi)\right\}\sin\varphi. \tag{45}$$

Equation (45) has a similar form as the current-phase relation Eq. (36) in the tunneling regime. For a short S layer with $L_S \ll \xi_S$, we find that the critical current is given by $I(\varphi = \pi/2)$, corresponding to the I_1 branch discussed in Sec. IV B. Thus the critical current decreases below T_c^* as

$$I_c = G\left\{\Delta'|\cos\chi| - \frac{\Delta^2(T)}{T_c^*}[c^{\mathrm{qp}}|\cos\chi| + 2c^{\mathrm{cond}}]\right\} \tag{46}$$

(except for angles $\chi \sim \pi/2$). As in the tunneling regime, this peculiar temperature dependence provides a clear signature of the competition between odd/odd- and odd/even-frequency couplings. By contrast, in the opposite regime, $L_S \gg \xi_S$, the temperature regime below T_c^*, where the I_1 branch dominates, shrinks to zero. Thus the critical current is given by $I(\varphi = \pi/2 + 0^+)$, corresponding to the I_2 branch discussed in Sec. IV B. We obtain

$$I_c = G\left\{\Delta'|\cos\chi|e^{-L_S/\xi_S} + 2c^{\mathrm{cond}}\frac{\Delta^2(T)}{T_c^*}\right\}. \tag{47}$$

Here, the quasiparticle and condensate contributions add up, which leads to an enhancement of I_c below T_c^*.

VI. CONCLUSION

Using circuit theory, we have proposed a simple model for the Green function g_T of an effective triplet odd-frequency superconducting reservoir S_T. Then, we have studied the coexistence of singlet even-frequency and triplet odd-frequency superconducting correlations in an $S_T/S/S_T$ Josephson junction.

We predict that the competition between odd/odd-frequency and odd/even-frequency Josephson couplings may be observed in a peculiar temperature dependence of the critical current of the $S_T/S/S_T$ junction below the transition temperature T_c^* of the central superconductor. For a large range of parameters, the critical current either increases very slowly or even decreases when lowering the temperature below T_c^*. This is in sharp contrast with a conventional $S'/S/S'$ junction, where the superconducting transition of the central superconductor leads to an enhancement of the critical current [23].

We propose to realize such an $S_T/S/S_T$ junction by fabricating a hybrid $S'/F'/F/S/F/F'/S'$ junction, i.e., by inserting a superconducting layer in the middle of an $S/F'/F/F'/S$ junction such as the ones presented in Refs. [5,6].

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APPENDIX: DERIVATION OF THE GREEN FUNCTION IN THE STRONG FERROMAGNET *F*

At large h, the solution of the equation

$$[h\sigma_z\tau_z + A, g] = 0, \tag{A1}$$

with $g^2 = 1$, can be expanded perturbatively in $1/h$: $g = g^{(0)} + (1/h)g^{(1)} + \cdots$. Let us now introduce the decomposition $A = A_\perp + A_\parallel$, with $[A_\parallel, \sigma_z\tau_z] = 0$ and $\{A_\perp, \sigma_z\tau_z\} = 0$. Similarly, $g^{(n)} = g_\parallel^{(n)} + g_\perp^{(n)}$.

In the leading order in h, Eq. (A1) yields $g_\perp^{(0)} = 0$, while the normalization condition reads $(g_\parallel^{(0)})^2 = 1$. In the next order, Eq. (A1) yields

$$2\sigma_z\tau_z g_\perp^{(1)} + [A, g_\parallel^{(0)}] = 0, \tag{A2}$$

while the normalization condition reads $\{g_\perp^{(1)} + g_\parallel^{(1)}, g_\parallel^{(0)}\} = 0$. It is solved with $g_\parallel^{(0)} = A_\parallel/\sqrt{A_\parallel^2}$ (note that $A_\parallel^2$ is scalar), $g_\perp^{(1)} = -(1/2)\sigma_z\tau_z[A_\perp, g_\parallel^{(0)}]$, and $g_\parallel^{(1)} = 0$.

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